Chapter 4

Continuous Random Variables

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- **4.1 Densities and probabilities**
- 4.2 Expectation of a single random variable

Definition

- We say that X is a **continuous random variable** if $\mathsf{P}(X \in B)$ has the form
$$\mathsf{P}(X \in B) = \int_B f(t)dt$$
- Usually, the set B is an interval such as $B = [a, b]$. In this case,
$$\mathsf{P}(a \le X \le b) = \int_a^b f(t)dt. \tag{1}$$
- A **probability density function** (pdf) is a nonnegative function that integrates to one.

Uniform distribution

- The simplest continuous random variable is the uniform.
- It is used to model experiments in which the outcome is constrained to lie in a known interval, say $[a, b]$, and all outcomes are equally likely.
- We write $f \sim \text{uniform}[a, b]$ if $a < b$ and

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

- This density is shown in Figure 1.

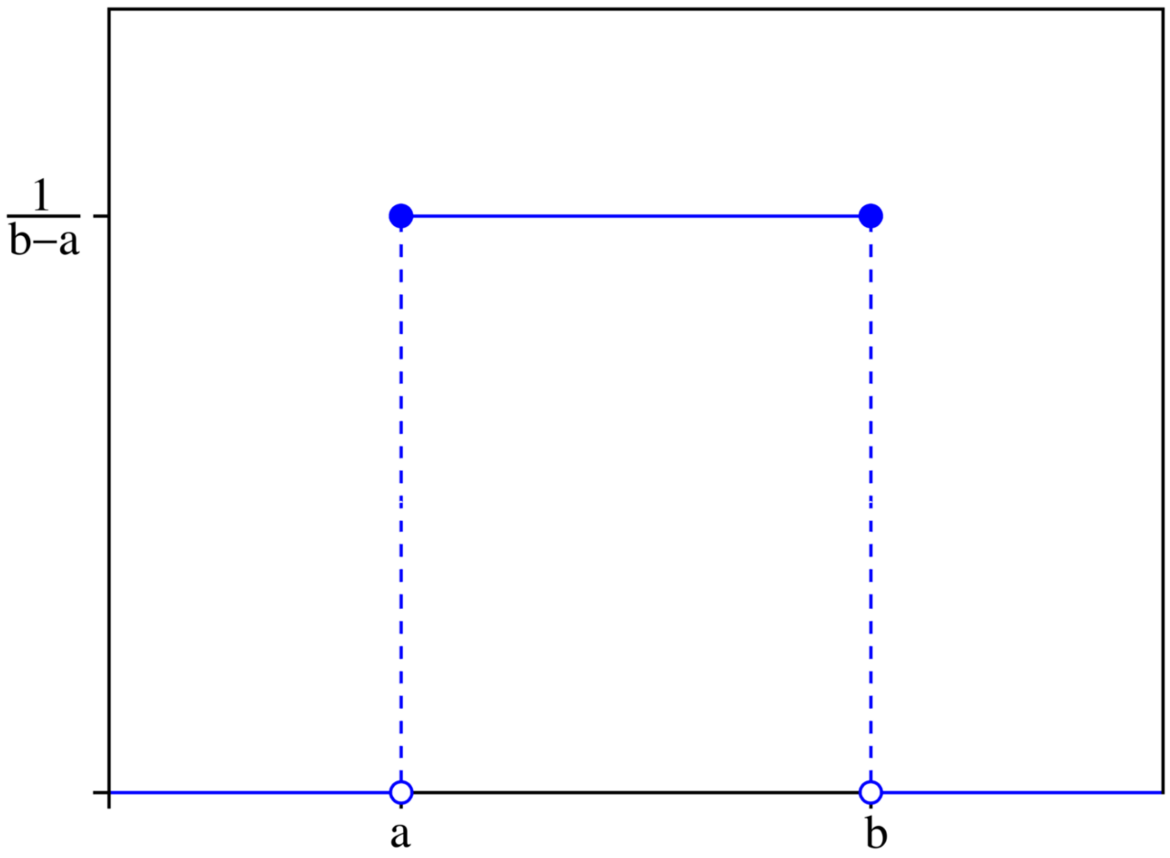

Figure 1: The uniform density on $[a, b]$.

Example 4.1

- In coherent radio communications, the phase difference between the transmitter and the receiver, denoted by Θ, is modelled as having a density $f \sim \text{uniform}[-\pi, \pi]$.
- Find $\mathsf{P}(\Theta \leq 0)$ and $\mathsf{P}(\Theta \leq \pi/2)$.

Solution

$$
\begin{aligned}
\mathsf{P}(\Theta \le 0) &= \int_{-\pi}^{0} f(\theta)d\theta = \int_{-\pi}^{0} \frac{1}{2\pi}d\theta = \frac{1}{2} \\
\mathsf{P}(\Theta \le \frac{\pi}{2}) &= \int_{-\pi}^{\pi/2} f(\theta)d\theta = \int_{-\pi}^{\pi/2} \frac{1}{2\pi}d\theta = \frac{3}{4}
\end{aligned}
$$

Exponential distribution

- Another simple continuous random variable is the exponential with parameter $\lambda > 0$.
- We write $f \sim \exp(\lambda)$ if

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

- Several exponential densities are in Figure 2.
- As λ increases, the height increases and the width decreases.
- It is easy to check that f integrates to one.

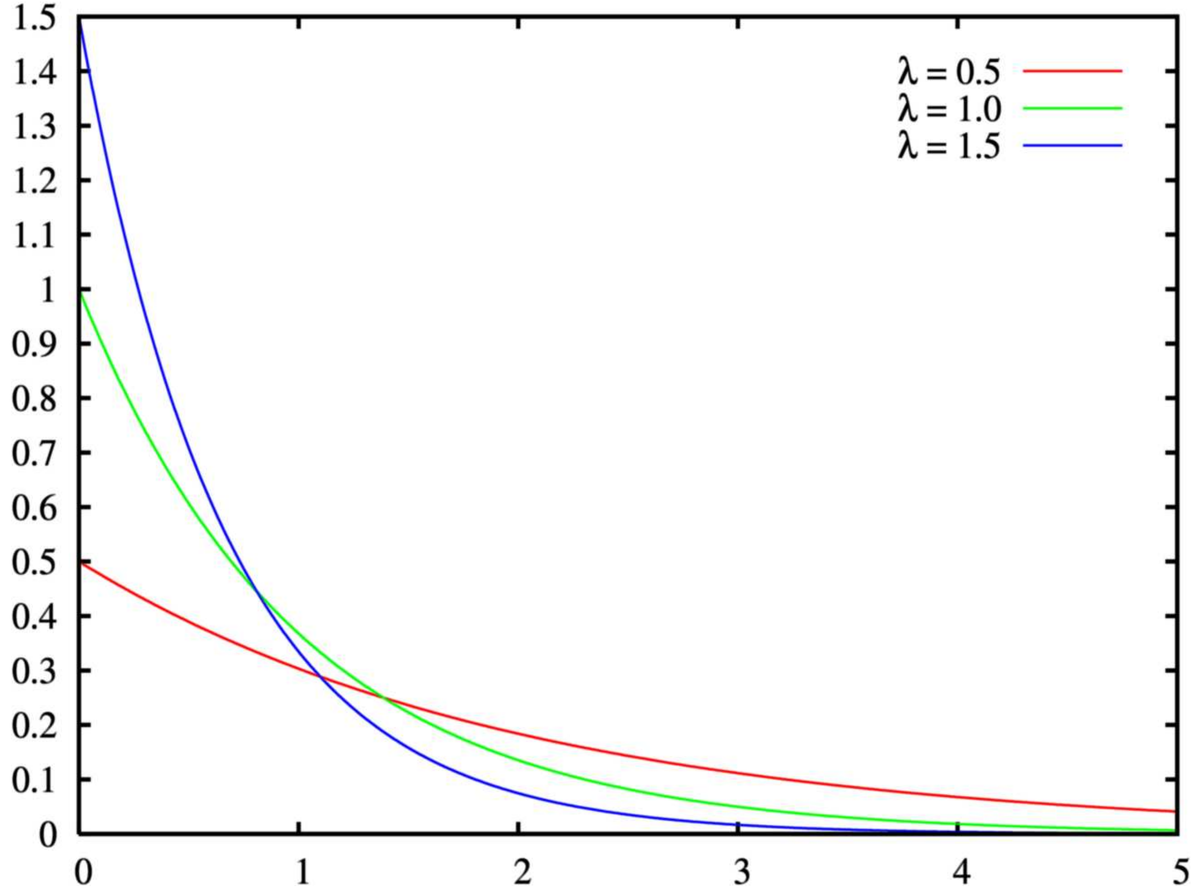

Figure 2: The exponential densities with different values of λ.

Laplace distribution

- Related to the exponential is the Laplace, sometimes called the double-sided exponential.
- For $b > 0$, we write $f \sim \text{Laplace}(\mu, b)$ if
$$f(x) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}.$$
- Several Laplace densities are shown in Figure 3.

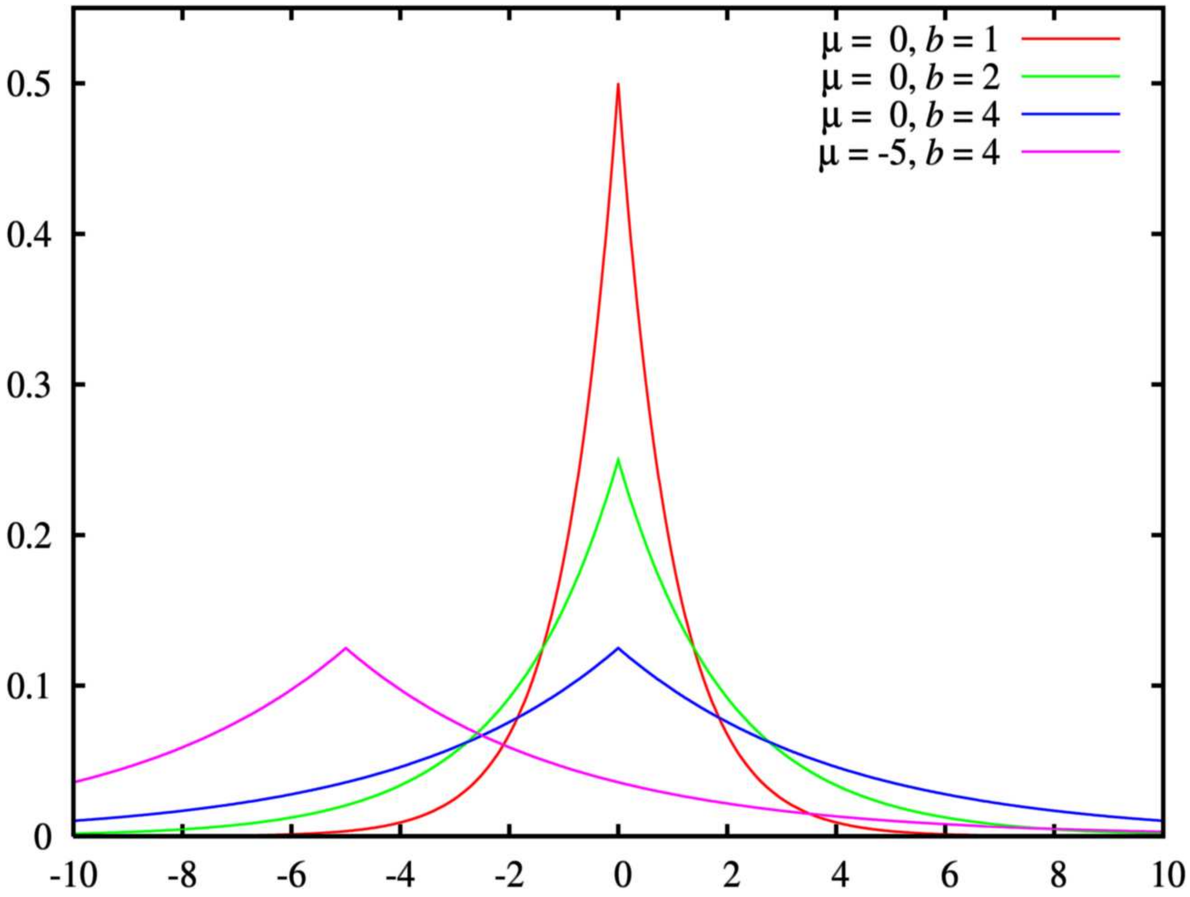

Figure 3: The Laplace densities with different values of μ and b.

Cauchy distribution

- The Cauchy random variable with parameter $\gamma > 0$ is also easy to work with.
- We write $f \sim \text{Cauchy}(x_0, \gamma)$ if

$$f(x) = \frac{1}{\pi} \frac{\gamma}{(x - x_0)^2 + \gamma^2}.$$

- Several Cauchy densities are shown in Figure 4.
- As γ increases, the height decreases and the width increases.

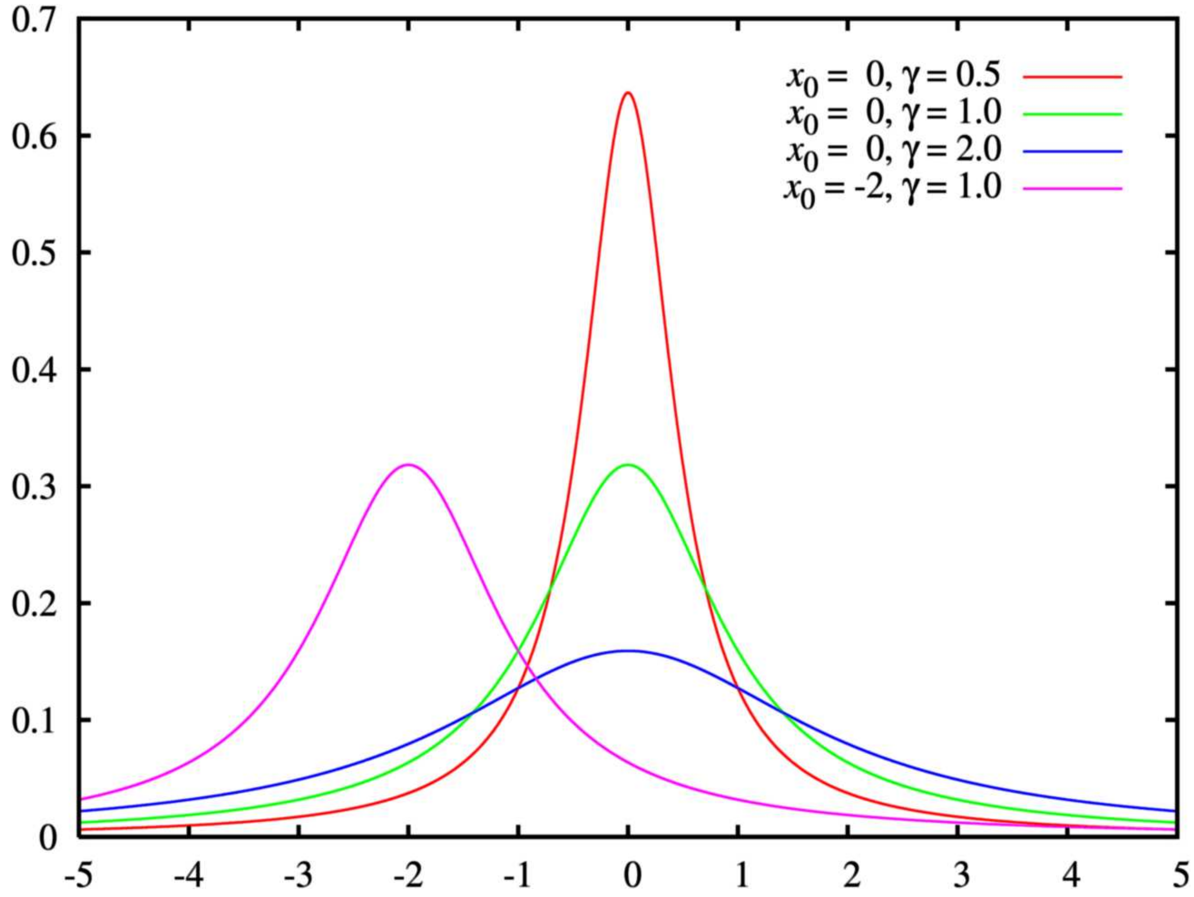

Figure 4: The Cauchy densities with different values of x_0 and γ.

Example 4.5

- In the $\lambda-$lottery you choose a number λ with $1 \leq \lambda \leq 10$.
- Then a random variable X is chosen according to the Cauchy density with parameter λ.
- If $|X| \geq 1$, then you win the lottery.
- Which value of λ should you choose to maximize your probability of winning?

Solution (1/2)

- Your probability of winning is

$$\begin{aligned}\mathsf{P}(|X| \geq 1) &= \mathsf{P}(X \geq 1 \texttt{ or } X \leq -1) \\ &= \int_1^\infty f(x)dx + \int_{-\infty}^{-1} f(x)dx,\end{aligned}$$

  where $f(x)$ is the Cauchy density.

- Since the Cauchy density is an even function,

$$\mathsf{P}(|X| \geq 1) = 2\int_1^\infty \frac{\lambda/\pi}{\lambda^2 + x^2}dx$$

Solution (2/2)

- Now make the change of variable $y = \dfrac{x}{\lambda}, dy = \dfrac{dx}{\lambda}$, to get

$$\mathsf{P}(|X| \geq 1) = 2 \int_{1/\lambda}^{\infty} \frac{1/\pi}{1+y^2} dy.$$

- Since the integrand is nonnegative, the integral is maximized by maximizing λ.

- Hence, choose $\lambda = 10$ maximizes your probability of winning.

Gaussian distribution

- The most important density is the Gaussian or normal.
- For $\sigma^2 > 0$, we write $f \sim N(\mu, \sigma^2)$ if

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big[-\frac{1}{2}(\frac{x-\mu}{\sigma})^2 \Big], \tag{2}$$

  where σ is the positive square root of σ^2.
- As σ increases, the height of the density decreases and it becomes wider as illustrated in Figure 5.

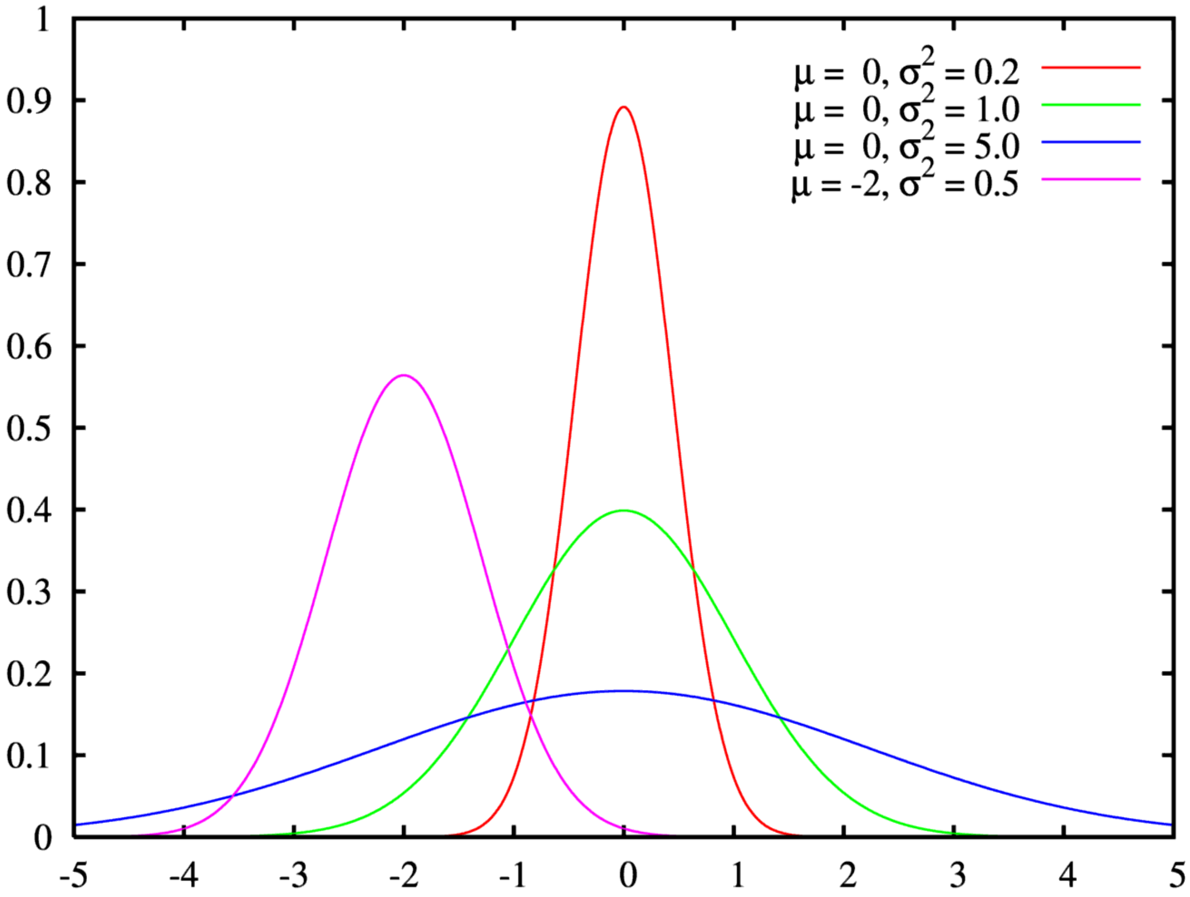

Figure 5: The normal densities with different values of μ and σ^2.

Homework

- Problems 1, 4(a), 5, 6, 7, 8.

- 4.1 Densities and probabilities
- **4.2 Expectation of a single random variable**

Expectation of a single random variable

- For a discrete random variable X with pmf p_X, we compute expectations using

$$\mathsf{E}[g(X)] = \sum_i g(x_i) p_X(x_i).$$

- Analogously, for a continuous random variable X with pdf f_X, we have

$$\mathsf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx. \tag{3}$$

Example 4.7

- If X is a uniform$[a, b]$ random variable, find $\mathsf{E}[X], \mathsf{E}[X^2]$, and $\mathsf{var}(X)$.

Solution (1/3)

- To find $\mathsf{E}[X]$, write

$$\mathsf{E}[X] = \int_a^b x f(x) dx = \int_a^b \frac{x}{b-a} dx = \frac{x^2}{2(b-a)}\Big|_a^b,$$

which simplifies to

$$\frac{b^2 - a^2}{2(b-a)} = \frac{(b+a)(b-a)}{2(b-a)} = \frac{(b+a)}{2}.$$

Solution (2/3)

- To compute the second moment, write

$$\mathsf{E}[X^2] = \int_a^b x^2 f(x)dx = \int_a^b \frac{x^2}{b-a}dx = \frac{x^3}{3(b-a)}\Big|_a^b,$$

which simplifies to

$$\frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ba+a^2)}{3(b-a)} = \frac{(b^2+ba+a^2)}{3}.$$

Solution (3/3)

- Since $\mathsf{var}(X) = \mathsf{E}[X^2] - (\mathsf{E}[X])^2$, we have

$$\begin{aligned}
\mathsf{var}(X) &= \frac{b^2 + ba + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\
&= \frac{b^2 - 2ba + a^2}{12} \\
&= \frac{(b-a)^2}{12}.
\end{aligned}$$

Homework

- Problems 23, 29, 30, 32, 33.