Chapter 3

More about Discrete Random Variables

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- **3.2 The binomial random variables**
- 3.4 Conditional probability
- 3.5 Conditional expectation

- In many problems, the quantity of interest can be expressed in the form $Y = X_1 + \cdots + X_n$, where the X_i are independent Berboulli(p) random variables.
- The random variable Y is called a **binomial**(n, p) random variable.
- Its probability mass function is

$$p_Y(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \ldots, n. \tag{1}$$

Example 3.6

- A sample of radioactive material is composed of n molecules.
- Each molecule has probability p of emitting an alpha particle, and the particles are emitted independently.
- Show that the number of particles emitted is a sum of independent Bernoulli random variables.

Solution

- Let $X_i = 1$ if the ith molecule emits an alpha particle, and $X_i = 0$ otherwise.
- Then the X_i are independent Bernoulli(p), and $Y = X_1 + \cdots + X_n$ counts the number of alpha particle emitted.

Homework

- Problems 11, 12, 13, 14.

- 3.2 The binomial random variables
- **3.4 Conditional probability**
- 3.5 Conditional expectation

Conditional probability mass functions

- For discrete random variables, we define the **conditional probability mass functions**,

$$\begin{aligned} p_{X|Y}(x_i|y_j) &:= \mathsf{P}(X = x_i | Y = y_j) \\ &= \frac{\mathsf{P}(X = x_i, Y = y_j)}{\mathsf{P}(Y = y_j)} = \frac{p_{XY}(x_i, y_j)}{p_Y(y_j)}, \end{aligned} \tag{2}$$

and

$$\begin{aligned} p_{Y|X}(y_j|x_i) &:= \mathsf{P}(Y = y_j | X = x_i) \\ &= \frac{\mathsf{P}(X = x_i, Y = y_j)}{\mathsf{P}(X = x_i)} = \frac{p_{XY}(x_i, y_j)}{p_X(x_i)}. \end{aligned} \tag{3}$$

Example 3.11

- To transmit message i using an optical communication system.
- When light of intensity λ_i strikes the photodetector, the number of photoelectrons generated is a Poisson(λ_i) random variable.
- Find the conditional probability that the number of photoelectrons observed at the photodetector is less than 2 given that message i was sent.

Solution

- Let X denote the message to be sent, and let Y denote the number of photoelectrons generated by the photodetector.
- The problem statement is telling us that

$$\mathsf{P}(Y = n | X = i) = \frac{{\lambda_i}^n e^{-\lambda_i}}{n!}, \quad n = 0, 1, 2, \ldots$$

- The conditional probability to be calculated is

$$\begin{aligned} \mathsf{P}(Y < 2 | X = i) &= \mathsf{P}(Y = 0 \text{ or } Y = 1 | X = i) \\ &= \mathsf{P}(Y = 0 | X = i) + \mathsf{P}(Y = 1 | X = i) \\ &= e^{-\lambda_i} + \lambda_i e^{-\lambda_i}. \end{aligned}$$

The law of total probability

- If Y is a discrete random variable taking the value y_j, then

$$\begin{aligned} \mathsf{P}(Y = y_j) &= \sum_i \mathsf{P}(Y = y_j | X = x_i) \mathsf{P}(X = x_i) \\ &= \sum_i p_{Y|X}(y_j | x_i) p_X(x_i). \end{aligned} \tag{4}$$

Example 3.15

- Radioactive samples give off alpha-particles at a rate based on the size of the sample.
- For a sample of size k, suppose that the number of particles observed is a Poisson random variable Y with parameter k.
- If the sample size is a geometric$_1(p)$ random variable X, find $\mathsf{P}(Y = 0)$ and $\mathsf{P}(X = 1|Y = 0)$.

Solution (1/3)

- The problem statement is telling us that $\mathsf{P}(Y = n|X = k)$ is the Possion pmf with parameter k.

$$\mathsf{P}(Y = n|X = k) = \frac{k^n e^{-k}}{n!}, \quad n = 0, 1, \ldots$$

- In particular, note that $\mathsf{P}(Y = 0|X = k) = e^{-k}$.

Solution (2/3)

- Now use the law of total probability to write

$$
\begin{aligned}
\mathsf{P}(Y=0) &= \sum_{k=1}^{\infty} \mathsf{P}(Y=0|X=k)\cdot \mathsf{P}(X=k) \\
&= \sum_{k=1}^{\infty} e^{-k}\cdot(1-p)p^{k-1} \\
&= \frac{1-p}{p}\sum_{k=1}^{\infty}\left(\frac{p}{e}\right)^{k} \\
&= \frac{1-p}{p}\frac{p/e}{1-p/e} = \frac{1-p}{e-p}.
\end{aligned}
$$

Solution (3/3)

- Next,

$$\begin{aligned}
\mathsf{P}(X=1|Y=0) &= \frac{\mathsf{P}(X=1, Y=0)}{\mathsf{P}(Y=0)} \\
&= \frac{\mathsf{P}(Y=0|X=1)\mathsf{P}(X=1)}{\mathsf{P}(Y=0)} \\
&= e^{-1} \cdot (1-p) \cdot \frac{e-p}{1-p} \\
&= 1 - \frac{p}{e}.
\end{aligned}$$

The substitution law

- It is often that Z is a function of X and Y, say $Z = g(X, Y)$, and we are interested in $\mathsf{P}(Z = z)$.

$$\begin{aligned} \mathsf{P}(Z = z) &= \sum_i \mathsf{P}(Z = z | X = x_i)\mathsf{P}(X = x_i) \\ &= \sum_i \mathsf{P}(g(X, Y) = z | X = x_i)\mathsf{P}(X = x_i). \end{aligned}$$

- We claim that

$$\mathsf{P}(g(X, Y) = z | X = x_i) = \mathsf{P}(g(x_i, Y) = z | X = x_i). \tag{5}$$

- This property is known as the **substitution law** of conditional probability.

Example 3.17

- A random, integer-valued signal X is transmitted over a channel subject to independent, additive, integer-valued noise Y.
- The received signal is $Z = X + Y$.
- To estimate X based on the received value Z, the system designer wants to use the conditional pmf $p_{X|Z}$.
- Find the desired conditional pmf.

Solution (1/3)

- Let X and Y be independent, discrete, integer-valued random variable with pmfs p_X and p_Y, respectively.

$$
\begin{aligned}
\mathsf{P}_{X|Z}(i|j) &= \mathsf{P}(X = i|Z = j) \\
&= \frac{\mathsf{P}(X = i, Z = j)}{\mathsf{P}(Z = j)} \\
&= \frac{\mathsf{P}(Z = j|X = i)\mathsf{P}(X = i)}{\mathsf{P}(Z = j)} \\
&= \frac{\mathsf{P}(Z = j|X = i)p_X(i)}{\mathsf{P}(Z = j)}
\end{aligned}
$$

Solution (2/3)

- Use the substitution law followed by independence to write

$$\begin{aligned}
\mathsf{P}(Z = j|X = i) &= \mathsf{P}(X + Y = j|X = i) \\
&= \mathsf{P}(i + Y = j|X = i) \\
&= \mathsf{P}(Y = j - i|X = i) \\
&= \frac{\mathsf{P}(Y = j - i, X = i)}{\mathsf{P}(X = i)} \\
&= \frac{\mathsf{P}(Y = j - i)\mathsf{P}(X = i)}{\mathsf{P}(X = i)} \\
&= \mathsf{P}(Y = j - i) = p_Y(j - i).
\end{aligned}$$

Solution (3/3)

- Use the law of total probability to compute

$$p_Z(j) = \sum_i \mathsf{P}(Z = j | X = i)\mathsf{P}(X = i) = \sum_i p_Y(j - i)p_X(i).$$

- It follows that

$$p_{X|Z}(i|j) = \frac{p_Y(j - i)p_X(i)}{\sum_k p_Y(j - k)p_X(k)}.$$

Homework

- Problems 23, 24, 27, 30, 31.

- 3.2 The binomial random variables
- 3.4 Conditional probability
- **3.5 Conditional expectation**

Conditional expectation

- Just as we develop expectation for discrete random variables, we can develop conditional expectation in the same way.
- This leads to the formula

$$\mathsf{E}[g[Y]|X = x_i] = \sum_j g(y_j) p_{Y|X}(y_j|x_i). \tag{6}$$

Example 3.19

- The random number Y of alpha particles emitted by a radioactive sample is conditionally Poisson(k) given that the sample size $X = k$.
- Find $\mathsf{E}[Y|X = k]$.

Solution

- We must compute

$$\mathsf{E}[Y|X=k]=\sum_n n\mathsf{P}(Y=n|X=k),$$

where

$$\mathsf{P}(Y=n|X=k)=\frac{k^n e^{-k}}{n!},\quad n=0,1,\ldots$$

- Hence,

$$\mathsf{E}[Y|X=k]=\sum_{n=0}^{\infty} n\frac{k^n e^{-k}}{n!}=k.$$

Substitution law for conditional expectation

- We call

$$\mathsf{E}[g(X,Y)|X = x_i] = \mathsf{E}[g(x_i,Y)|X = x_i] \tag{7}$$

the **substitution law for conditional expectation**.

Law of total probability for expectation

- In Section 3.4 we discuss the law of total probability, which shows how to compute probabilities in terms of conditional probabilities.

- We now derive the analogous formula for expectation.

$$
\begin{aligned}
& \sum_i \mathsf{E}[g(X,Y)|X=x_i]p_X(x_i) \\
= & \sum_i \mathsf{E}[g(x_i,Y)|X=x_i]p_X(x_i) \\
= & \sum_i \Big[\sum_j g(x_i,y_j)p_{Y|X}(y_j|x_i)\Big]p_X(x_i) \\
= & \sum_i \sum_j g(x_i,y_j)p_{XY}(x_i,y_j) \\
= & \mathsf{E}[g(X,Y)]
\end{aligned}
$$

- Hence, the **law of total probability for expectation** is

$$\mathsf{E}[g(X,Y)] = \sum_i \mathsf{E}[g(X,Y)|X = x_i]p_X(x_i). \tag{8}$$

Homework

- Problems 40, 41, 42.