Numerical Analysis
Quiz 4: Least-Squares Fitting of a Curve to Data

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1. (40 %) The function $y = x/(c_1 + c_2 x)$ can be transformed into a linear relationship $z = c_1 w + c_2$ with the change of variables $z = 1/y, w = 1/x$. Write a MATLAB function that calls `linefit` to fit data to $y = x/(c_1 + c_2 x)$.

   Answer:

   $z = 1./y;$

   $w = 1./x;$

   $c = \texttt{linefit}(w, z);$

2. (60 %) Least-squares fitting problems with only one undetermined coefficient lead to particularly simple computational procedures. Derive the equations for finding the coefficient c of the following equations.

   (a) $y = cx$
   (b) $y = cx^2$
   (c) $y = x^c$

   Constraint: You are not allowed to use the normal equation directly.
   Hint: Set the derivative of $\rho = \sum r_i^2$ equal to zero.

   Answer:

$$
\begin{aligned}
&\mathbf{A}c = \mathbf{y} \\
&\mathbf{r} = \mathbf{y} - \mathbf{A}c \\
&\rho = \sum r_i^2 = \mathbf{r}^T\mathbf{r} = (\mathbf{y} - \mathbf{A}c)^T(\mathbf{y} - \mathbf{A}c) \\
&\quad = \mathbf{y}^T\mathbf{y} - \mathbf{A}^T c\mathbf{y} - \mathbf{y}^T\mathbf{A}c + \mathbf{A}^T\mathbf{A}c^2 \\
&\quad = \mathbf{y}^T\mathbf{y} - 2c\mathbf{A}^T\mathbf{y} + c^2\mathbf{A}^T\mathbf{A} \\
&\Rightarrow \frac{\partial \rho}{\partial c} = -2\mathbf{A}^T\mathbf{y} + 2c\mathbf{A}^T\mathbf{A} = 0 \\
&\Rightarrow 2c\mathbf{A}^T\mathbf{A} = 2\mathbf{A}^T\mathbf{y} \\
&\Rightarrow c = \frac{\mathbf{A}^T\mathbf{y}}{\mathbf{A}^T\mathbf{A}}
\end{aligned}
$$

(a)

$$
\mathbf{A} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, c = \frac{\mathbf{A}^T\mathbf{y}}{\mathbf{A}^T\mathbf{A}}
$$

(b)

$$
\mathbf{A} = \begin{bmatrix} (x_1)^2 \\ (x_2)^2 \\ \vdots \\ (x_m)^2 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, c = \frac{\mathbf{A}^T\mathbf{y}}{\mathbf{A}^T\mathbf{A}}
$$

(c)

$$
\begin{aligned}
&\mathbf{y} = \mathbf{x}^c \\
&\log \mathbf{y} = c \log \mathbf{x}
\end{aligned}
$$

$$
\mathbf{A} = \begin{bmatrix} \log x_1 \\ \log x_2 \\ \vdots \\ \log x_m \end{bmatrix}, \mathbf{y} = \begin{bmatrix} \log y_1 \\ \log y_2 \\ \vdots \\ \log y_m \end{bmatrix}, c = \frac{\mathbf{A}^T\mathbf{y}}{\mathbf{A}^T\mathbf{A}}
$$