Chapter 16 Simplex Method

An Introduction to Optimization

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Solving Linear Equations Using Row Operations

- An elementary row operation on a given matrix is an algebraic manipulation of the matrix that corresponds to one of the following
  - 1. Interchanging any two rows of the matrix
  - 2. Multiplying one of its rows by a real nonzero number.
  - 3. Adding a scalar multiple of one row to another row.
- An elementary row operation on a matrix is equivalent to premultiplying the matrix by a corresponding ***elementary matrix***.
- Definition 16.1: We call $\boldsymbol{E}$ an ***elementary matrix of the first kind*** if $\boldsymbol{E}$ is obtained from the identity matrix $\boldsymbol{I}$ by interchanging any two of its row. Note that $\boldsymbol{E} = \boldsymbol{E}^{-1}$

Solving Linear Equations Using Row Operations

- Definition 16.2: We call $\boldsymbol{E}$ an ***elementary matrix of the second kind*** if $\boldsymbol{E}$ is obtained from the identity matrix $\boldsymbol{I}$ by multiplying one of its rows by a real number $\alpha \neq 0$
- Definition 16.3: We call $\boldsymbol{E}$ an ***elementary matrix of the third kind*** if $\boldsymbol{E}$ is obtained from the identity matrix $\boldsymbol{I}$ by adding β times one row to another row of $\boldsymbol{I}$.
- Definition 16.4: An ***elementary row operation*** on a given matrix is a premultiplication of the given matrix by a corresponding elementary matrix of the respective kind.

Solving Linear Equations Using Row Operations

- Because elementary matrices are invertible, we can define the corresponding inverse elementary row operations. Consider a system of n linear equations in n unknowns $x_1, ..., x_n$ with right-hand sides $b_1, ..., b_n$. In matrix form this system may be written as $\boldsymbol{Ax} = \boldsymbol{b}$, where

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \qquad \boldsymbol{A} \in R^{n \times n}$$

- If $\boldsymbol{A}$ is invertible, then $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$. We now show that $\boldsymbol{A}^{-1}$ can be computed effectively using elementary row operations.

Solving Linear Equations Using Row Operations

- Theorem 16.1: Let $\boldsymbol{A} \in R^{n\times n}$ be a given matrix. Then, $\boldsymbol{A}$ is nonsingular (invertible) if and only if there exist elementary matrices $\boldsymbol{E}_i, i = 1, ..., t$ such that
$$\boldsymbol{E}_t \cdots \boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{I}$$
- We first form an augmented matrix $[\boldsymbol{A}, \boldsymbol{I}]$, and then apply elementary row operations so that $\boldsymbol{A}$ is transformed into $\boldsymbol{I}$; that is, we obtain
$$\boldsymbol{E}_t \cdots \boldsymbol{E}_2\boldsymbol{E}_1[\boldsymbol{A}, \boldsymbol{I}] = [\boldsymbol{I}, \boldsymbol{B}]$$
It then follows that $\boldsymbol{B} = \boldsymbol{E}_t \cdots \boldsymbol{E}_2\boldsymbol{E}_1 = \boldsymbol{A}^{-1}$

Solving Linear Equations Using Row Operations

- Let $\boldsymbol{A}^{-1} = \boldsymbol{E}_t \cdots \boldsymbol{E}_2 \boldsymbol{E}_1$, thus $\boldsymbol{E}_t \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} \boldsymbol{x} = \boldsymbol{E}_t \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{b}$ and hence, $\boldsymbol{x} = \boldsymbol{E}_t \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{b}$
- For an augmented matrix $[\boldsymbol{A}, \boldsymbol{b}]$. Then, perform a sequence of row elementary operations on this augmented matrix until we obtain $[\boldsymbol{I}, \tilde{\boldsymbol{b}}]$. From the above we have that if $\boldsymbol{x}$ is a solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, then it is also a solution to $\boldsymbol{E}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{E}\boldsymbol{b}$, where $\boldsymbol{E} = \boldsymbol{E}_t \cdots \boldsymbol{E}_2 \boldsymbol{E}_1$ represents a sequence of elementary row operations. Because $\boldsymbol{E}\boldsymbol{A} = \boldsymbol{I}$, and $\boldsymbol{E}\boldsymbol{b} = \tilde{\boldsymbol{b}}$, it follows that $\boldsymbol{x} = \tilde{\boldsymbol{b}}$ is the solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{A} \in R^{n \times n}$ invertible.

Solving Linear Equations Using Row Operations

- Suppose now that $\boldsymbol{A} \in R^{m \times n}$ where $m < n$, and $rank(\boldsymbol{A}) = m$. Then, $\boldsymbol{A}$ is not a square matrix. Clearly, in this case the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ has infinitely many solutions. Without loss of generality, we can assume that the first m columns of $\boldsymbol{A}$ are linearly independent. Then, if we perform a sequence of elementary row operations on the augmented matrix $[\boldsymbol{A}, \boldsymbol{b}]$ as before, we obtain $[\boldsymbol{I}, \boldsymbol{D}, \tilde{\boldsymbol{b}}]$, where $\boldsymbol{D}$ is an $m \times (n-m)$ matrix.
- Let $\boldsymbol{x} \in R^n$ be a solution to $\boldsymbol{Ax} = \boldsymbol{b}$ and write $\boldsymbol{x} = [\boldsymbol{x}_B^T, \boldsymbol{x}_D^T]^T$, where $\boldsymbol{x}_B \in R^m$, $\boldsymbol{x}_D \in R^{(n-m)}$. Then, $[\boldsymbol{I}, \boldsymbol{D}]\boldsymbol{x} = \tilde{\boldsymbol{b}}$, which we can rewrite as $\boldsymbol{x}_B + \boldsymbol{D}\boldsymbol{x}_D = \tilde{\boldsymbol{b}}$ or $\boldsymbol{x}_B = \tilde{\boldsymbol{b}} - \boldsymbol{D}\boldsymbol{x}_D$. Note that for an arbitrary $\boldsymbol{x}_D \in R^{(n-m)}$, if $\boldsymbol{x}_B = \tilde{\boldsymbol{b}} - \boldsymbol{D}\boldsymbol{x}_D$, then the resulting vector $\boldsymbol{x} = [\boldsymbol{x}_B^T, \boldsymbol{x}_D^T]^T$ is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Solving Linear Equations Using Row Operations

- In particular, $[\tilde{\boldsymbol{b}}^T, \mathbf{0}^T]^T$ is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$. We often refer to the basic solution $[\tilde{\boldsymbol{b}}^T, \mathbf{0}^T]^T$ as a ***particular solution*** to $\boldsymbol{Ax} = \boldsymbol{b}$
- Note that $[-(\boldsymbol{Dx}_D)^T, \boldsymbol{x}_D^T]^T$ is a solution to $\boldsymbol{Ax} = \mathbf{0}$. Any solution to $\boldsymbol{Ax} = \boldsymbol{b}$ has the form

$$\boldsymbol{x} = \begin{bmatrix} \tilde{\boldsymbol{b}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} -\boldsymbol{Dx}_D \\ \boldsymbol{x}_D \end{bmatrix}$$

for some $\boldsymbol{x}_D \in R^{(n-m)}$

The Canonical Augmented Matrix

- Consider the system of simultaneous linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ $rank(\boldsymbol{A}) = m$. Using a sequence of elementary row operations and reordering the variables if necessary, we transform the system $\boldsymbol{Ax} = \boldsymbol{b}$ into the following ***canonical form***:

$$\begin{aligned}
x_1 \qquad\qquad\qquad\qquad & +y_{1m+1}x_{m+1} + \cdots + y_{1n}x_n = y_{10} \\
x_2 \qquad\qquad\qquad & +y_{2m+1}x_{m+1} + \cdots + y_{2n}x_n = y_{20} \\
\ddots \qquad\qquad & \\
x_m & +y_{mm+1}x_{m+1} + \cdots + y_{mn}x_n = y_{m0}
\end{aligned}$$

This can be represented in matrix notation as

$$[\boldsymbol{I}_m, \boldsymbol{Y}_{m,n-m}]\boldsymbol{x} = \boldsymbol{y}_0$$

The Canonical Augmented Matrix

- Definition 16.5: A system $\boldsymbol{Ax} = \boldsymbol{b}$ is said to be in ***canonical form*** if among the n variables there are m variables with the property that each appears in only one equation, and its coefficient in that equation is unity.
- A system is in canonical form if by some reordering of the equations and the variables it takes the form $[\boldsymbol{I}_m, \boldsymbol{Y}_{m,n-m}]\boldsymbol{x} = \boldsymbol{y}_0$ If a system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ is not in canonical form, we can transform the system into canonical form by a sequence of elementary row operations. The system in canonical form has the same solution as the original system and is called the ***canonical representation*** of the system with respect to the basis $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$

The Canonical Augmented Matrix

- There are, in general, many canonical representations of a given system, depending on which columns of $\boldsymbol{A}$ we transform into the columns of $\boldsymbol{I}_m$. We call the augmented matrix $[\boldsymbol{I}_m, \boldsymbol{Y}_{m,n-m}, \boldsymbol{y}_0]$ of the canonical representation of a given system the ***canonical augmented matrix*** of the system with respect to the basis $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$. Of course, there may be many canonical augmented matrices of a given system, depending on which columns of $\boldsymbol{A}$ are chosen as basic columns.

The Canonical Augmented Matrix

- The variables corresponding to basic columns in a canonical representation of a given system are the basic variables, whereas the other variables are the nonbasic variables. For $[\boldsymbol{I}_m, \boldsymbol{Y}_{m,n-m}]\boldsymbol{x} = \boldsymbol{y}_0$, the variables $x_1, \ldots, x_m$ are the basic variables and the other variables are the nonbasic variables.
- Note that in general the basic variables need not be the first m variables. However, for convenience and without loss of generality, the basic variables are assumed so.
- Having done so, the corresponding basic solution is

$$\begin{aligned} x_1 &= y_{10} \\ &\vdots \\ x_m &= y_{m0} \\ x_{m+1} &= 0 \\ &\vdots \\ x_n &= 0 \end{aligned} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{y}_0 \\ \boldsymbol{0} \end{bmatrix}$$

The Canonical Augmented Matrix

- Given a system of equations $\boldsymbol{Ax} = \boldsymbol{b}$, consider the associated canonical augmented matrix

$$[\boldsymbol{I}_m, \boldsymbol{Y}_{m,n-m}, \boldsymbol{y}_0] = \begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1m+1} & \cdots & y_{1n} & y_{10} \\ 0 & 1 & \cdots & 0 & y_{2m+1} & \cdots & y_{2n} & y_{20} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_{mm+1} & \cdots & y_{mn} & y_{m0} \end{bmatrix}$$

  From the augments above we conclude that

$$\boldsymbol{b} = y_{10}\boldsymbol{a}_1 + y_{20}\boldsymbol{a}_2 + \cdots + y_{m0}\boldsymbol{a}_m$$

- In other words, the entries in the last column of the canonical augmented matrix are the coordinates of the vector $\boldsymbol{b}$ with respect to the basis $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$

The Canonical Augmented Matrix

- The entries of all the other columns of the canonical augmented matrix have a similar interpretation. Specifically, the entries of the jth columns of the canonical augmented matrix, $j = 1, \ldots, n$ are the coordinates of $\boldsymbol{a}_j$ with respect to the basis $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$
- To see this, note that the first m columns of the augmented matrix form a basis (the standard basis). Every other vector in the augmented matrix can be expressed as a linear combination of these basis vectors by reading the coefficients down the corresponding column.

The Canonical Augmented Matrix

- Specifically, let $\boldsymbol{a}'_i, i = 1, ..., n+1$ be the i th column in the augmented matrix above. Clearly, since $\boldsymbol{a}'_1, ..., \boldsymbol{a}'_m$ form the standard basis, then for $m < j \leq n$

$$\boldsymbol{a}'_j = y_{1j}\boldsymbol{a}'_1 + y_{2j}\boldsymbol{a}'_2 + \cdots + y_{mj}\boldsymbol{a}'_m$$

Let $\boldsymbol{a}_i, i = 1, ..., n$ be the i th column of $\boldsymbol{A}$, and $\boldsymbol{a}_{n+1} = \boldsymbol{b}$. Now, $\boldsymbol{a}'_i = \boldsymbol{E}\boldsymbol{a}_i, i = 1, ..., n+1$, where $\boldsymbol{E}$ is a nonsingular matrix that represents the elementary row operations needed to transform $[\boldsymbol{A}, \boldsymbol{b}]$ into $[\boldsymbol{I}_m, \boldsymbol{Y}_{m,n-m}, \boldsymbol{y}_0]$. Therefore, for $m < j \leq n$, we also have

$$\boldsymbol{a}_j = y_{1j}\boldsymbol{a}_1 + y_{2j}\boldsymbol{a}_2 + \cdots + y_{mj}\boldsymbol{a}_m$$

Updating the Augmented Matrix

- Suppose that we are given the canonical representation of a system $\boldsymbol{Ax} = \boldsymbol{b}$. If we replace a basic variable by a nonbasic variable, what is the new canonical representation corresponding to the new set of basic variables? Specifically, suppose that we wish to replace the basis vector $\boldsymbol{a}_p, 1 \leq p \leq m$ by the vector $\boldsymbol{a}_q, m < q \leq n$. Provided that the first m vectors with $\boldsymbol{a}_p$ replaced by $\boldsymbol{a}_q$ are linearly independent, these vectors constitute a basis and every vector can be expressed as a linear combination of the new basic columns.

Updating the Augmented Matrix

- Let us now find the coordinates of the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ with respect to the new basis. These coordinates form the entries of the canonical augmented matrix of the system with respect to the new basis. In terms of the old basis, we can express $\boldsymbol{a}_q$ as

$$\boldsymbol{a}_q = \sum_{i=1}^{m} y_{iq}\boldsymbol{a}_i = \sum_{\substack{i=1\\ i\neq p}}^{m} y_{iq}\boldsymbol{a}_i + y_{pq}\boldsymbol{a}_p$$

- Note that the set of vectors $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{p-1}, \boldsymbol{a}_q, \boldsymbol{a}_{p+1}, \ldots, \boldsymbol{a}_m\}$ is linearly independent if and only if $y_{pq} \neq 0$. Solving the equation above for $\boldsymbol{a}_p$, we get

$$\boldsymbol{a}_p = \frac{1}{y_{pq}}\boldsymbol{a}_q - \sum_{\substack{i=1\\ i\neq p}}^{m} \frac{y_{iq}}{y_{pq}}\boldsymbol{a}_i$$

Updating the Augmented Matrix

- Recall that in terms of the old augmented matrix, any vector $\boldsymbol{a}_j, m < j \leq n$ can be expressed as

$$\boldsymbol{a}_j = y_{1j}\boldsymbol{a}_1 + y_{2j}\boldsymbol{a}_2 + \cdots + y_{mj}\boldsymbol{a}_m$$

Combining the last two equations yields

$$\boldsymbol{a}_j = \sum_{\substack{i=1 \\ i \neq p}}^{m} \left(y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq} \right) \boldsymbol{a}_i + \frac{y_{pj}}{y_{pq}} \boldsymbol{a}_q$$

- Denoting the entries of the new augmented matrix by y'_{ij}, we obtain $y'_{ij} = y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq}, i \neq p$ $\qquad y'_{pj} = \frac{y_{pj}}{y_{pq}}$

Therefore, the entries of the new canonical augmented matrix can be obtained from the entries of the old canonical augmented matrix via the formulas above. These equations are often called the ***pivot equations***, and y_{pq} , the ***pivot element***.

Updating the Augmented Matrix

- We refer to the operation on a given matrix by the formulas above as ***pivoting about the*** (p, q)***-th element.*** Note that pivoting about the (p, q)th element results in a matrix whose qth column has all zero entries, except the (p, q)th entry, which is unity.
- The pivoting operation can be accomplished via a sequence of elementary row operations, as was done in the proof of Theorem 16.1.

The Simplex Algorithm

- The essence of the simplex algorithm is to move from one basic feasible solution to another until an optimal basic feasible solution is found.
- Suppose that we are given the basic feasible solution

$$\boldsymbol{x} = [x_1, ..., x_m, 0, ..., 0]^T \qquad x_i \geq 0, i = 1, ..., m$$

  or equivalently

$$x_1 \boldsymbol{a}_1 + \cdots + x_m \boldsymbol{a}_m = \boldsymbol{b}$$

- In the simplex method we want to move from one basic feasible solution to another. This means that we want to change basic columns in such as way that the last column of the canonical augmented matrix remains nonnegative.

The Simplex Algorithm

- We assume that every basic feasible solution of $\boldsymbol{Ax} = \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}$ is a nondegenerate basic feasible solution. We make this assumption primarily for convenience – all arguments can be extended to include degeneracy.

The Simplex Algorithm

- Let us start with the basic columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, and assume that the corresponding basic solution $\boldsymbol{x} = [y_{10}, \ldots, y_{m0}, 0, \ldots, 0]^T$ is feasible; that is, the entries $y_{i0}, i = 1, \ldots, m$, in the last column of the canonical augmented matrix are positive.
- Suppose that we now decide to make the vector $\boldsymbol{a}_q$, $q > m$, a basic column. We first represent $\boldsymbol{a}_q$ in terms of the current basis as $\boldsymbol{a}_q = y_{1q}\boldsymbol{a}_1 + y_{2q}\boldsymbol{a}_2 + \cdots + y_{mq}\boldsymbol{a}_m$. Multiplying the above by $\epsilon > 0$ yields

$$\epsilon\boldsymbol{a}_q = \epsilon y_{1q}\boldsymbol{a}_1 + \epsilon y_{2q}\boldsymbol{a}_2 + \cdots + \epsilon y_{mq}\boldsymbol{a}_m$$

We combine this equation with $y_{10}\boldsymbol{a}_1 + \cdots + y_{m0}\boldsymbol{a}_m = \boldsymbol{b}$ to get

$$(y_{10} - \epsilon y_{1q})\boldsymbol{a}_1 + (y_{20} - \epsilon y_{2q})\boldsymbol{a}_2 + \cdots + (y_{m0} - \epsilon y_{mq})\boldsymbol{a}_m + \epsilon\boldsymbol{a}_q = \boldsymbol{b}$$

The Simplex Algorithm

- Note that the vector

$$\begin{bmatrix} y_{10} - \epsilon y_{1q} \\ \vdots \\ y_{m0} - \epsilon y_{mq} \\ 0 \\ \vdots \\ \epsilon \\ \vdots \\ 0 \end{bmatrix}$$

where ϵ appears in the qth position, is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$ If $\epsilon = 0$, then we obtain the old basic feasible solution. As ϵ is increased from zero, the qth component of the vector above increases. All other entries of this vector will increase or decrease linearly as ϵ is increased, depending on whether the corresponding y_{iq} is negative or positive.

The Simplex Algorithm

- For small enough ϵ , we have a feasible but nonbasic solution. If any of the components decreases as ϵ increases, we choose ϵ to be the smallest value where one (or more) of the components vanishes. That is,

$$\epsilon = \min_i \{ y_{i0}/y_{iq} : y_{iq} > 0 \}$$

- With this choice of ϵ we have a new basic feasible solution, with the vector $\boldsymbol{a}_q$ replacing $\boldsymbol{a}_p$, where p corresponds to the minimizing index $p = \arg\min_i \{ y_{i0}/y_{iq} : y_{iq} > 0 \}$. So, we now have a new basis $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{p-1}, \boldsymbol{a}_{p+1}, \dots, \boldsymbol{a}_m, \boldsymbol{a}_q$.
- As we can see, $\boldsymbol{a}_p$ was replaced by $\boldsymbol{a}_q$ in the new basis. We say that $\boldsymbol{a}_q$ ***enters*** the basis and $\boldsymbol{a}_p$ ***leaves*** the basis.

The Simplex Algorithm

- If the minimum in $\min_i\{y_{i0}/y_{iq} : y_{iq} > 0\}$ is achieved by more than a single index, then the new solution is degenerate and any of the zero components can be regarded as the component corresponding to the basic column that leaves the basis.
- If none of the y_{iq} are positive, then all components in the vector $[y_{10} - \epsilon y_{1q}, y_{20} - \epsilon y_{2q}, ..., y_{m0} - \epsilon y_{mq}, 0, ..., \epsilon, ..., 0]^T$ increase (or remain constant) as ϵ is increased, and no new basic feasible solution is obtained, no matter how large we make ϵ .
- In this case there are feasible solutions having arbitrarily large components, which means that the set Ω of feasible solutions is unbounded.

The Simplex Algorithm

- So far, we have discussed how to change from one basis to another, while preserving feasibility of the corresponding basic solution assuming that we have already chosen a nonbasic column to enter the basis. To complete our development of the simplex method, we need to consider two more issues.
- The first issue concerns the choice of which nonbasic column should enter the basis.
- The second issue is to find a stopping criterion, that is, a way to determine if a basic feasible solution is optimal or is not.

The Simplex Algorithm

- Suppose that we have fond a basic feasible solution. The main idea of the simplex method is to move from one basic feasible solution (extreme point of the set Ω) to another basic feasible solution at which the value of the objective function is smaller.
- Because there is only a finite number of extreme points of the feasible set, the optimal point will be reached after a finite number of steps.

The Simplex Algorithm

- We already know how to move from one extreme point of the set Ω to a neighboring one by updating the canonical augmented matrix. To see which neighboring solution we should move to and when to stop moving, consider the following basic feasible solution:

$$[\boldsymbol{x}_B^T, \mathbf{0}^T]^T = [y_{10}, ..., y_{m0}, 0, ..., 0]^T$$

together with the corresponding canonical augmented matrix, having an identity matrix appearing in the first m columns. The value of the objective function for any solution $\boldsymbol{x}$ is

$$z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

For our basic solution, the value of the objective function is

$$z = z_0 = \boldsymbol{c}_B^T \boldsymbol{x}_B = c_1 y_{10} + \cdots + c_m y_{m0}$$

where $\boldsymbol{c}_B^T = [c_1, c_2, ..., c_m]$

The Simplex Algorithm

- To see how the value of the objective function changes when we move from one basic feasible solution to another, suppose that we choose the qth column, $m < q \leq n$, to enter the basis.
- To update the canonical augmented matrix, let $p = \arg\min_i \{y_{i0}/y_{iq} : y_{iq} > 0\}$ and $\epsilon = y_{p0}/y_{pq}$. The new basic feasible solution is

$$\begin{bmatrix} y_{10} - \epsilon y_{1q} \\ \vdots \\ y_{m0} - \epsilon y_{mq} \\ 0 \\ \vdots \\ \epsilon \\ \vdots \\ 0 \end{bmatrix}$$

The Simplex Algorithm

- Note that the single ϵ appears in the qth component, whereas the pth component is zero. Observe that we would have arrived at the basic feasible solution above simply by updating the canonical augmented matrix using the pivot equations from the previous section

$$y'_{ij} = y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq}, i \neq p \qquad y'_{pj} = \frac{y_{pj}}{y_{pq}}$$

  where the qth column enters the basis and the pth column leaves [i.e., we pivot about the (p, q) th component]. The values of the basic variables are entries in the last column of the updated canonical augmented matrix.

The Simplex Algorithm

- The cost for this new basic feasible solution is

$$\begin{aligned} z &= c_1(y_{10} - y_{1q}\epsilon) + \cdots + c_m(y_{m0} - y_{mq}\epsilon) + c_q\epsilon \\ &= z_0 + [c_q - (c_1 y_{1q} + \cdots + c_m y_{mq})]\epsilon \end{aligned}$$

  where $z_0 = c_1 y_{10} + \cdots + c_m y_{m0}$. Let $z_q = c_1 y_{1q} + \cdots + c_m y_{mq}$, then $z = z_0 + (c_q - z_q)\epsilon$. Thus, if $z - z_0 = (c_q - z_q)\epsilon < 0$, then the objective function value at the new basic feasible solution above is smaller than the objective function value at the original solution (i.e., $z < z_0$). Therefore, if $c_q - z_q < 0$, then the new basic feasible solution with $\boldsymbol{a}_q$ entering the basis has a lower objective function value.

The Simplex Algorithm

- On the other hand, if the given basic feasible solution is such that for all $q = m+1, ..., n,\ c_q - z_q \geq 0$, then we can show that this solution is in fact an optimal solution.
- To show this, recall that any solution to $\boldsymbol{Ax} = \boldsymbol{b}$ can be represented as

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{y}_0 \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} -\boldsymbol{Y}_{m,n-m}\boldsymbol{x}_D \\ \boldsymbol{x}_D \end{bmatrix}$$

for some $\boldsymbol{x}_D = [x_{m+1}, ..., x_n]^T \in R^{(n-m)}$. Using manipulations similar to the above, we obtain

$$\boldsymbol{c}^T\boldsymbol{x} = z_0 + \sum_{i=m+1}^{n} (c_i - z_i) x_i$$

where $z_i = c_1 y_i + \cdots + c_m y_{mi}, i = m+1, ..., n$. For a feasible solution we have $x_i \geq 0, i = 1, ..., n$. Therefore, if $c_i - z_i \geq 0$ for all $i = m+1, ..., n$, then any feasible solution $\boldsymbol{x}$ will have objective function value $\boldsymbol{c}^T\boldsymbol{x}$ no smaller than z_0

The Simplex Algorithm

- Let $r_i = 0$ for $i = 1, ..., m$ and $r_i = c_i - z_i$ for $i = m+1, ..., n$ we call r_i the ith ***reduced cost coefficient*** or ***relative cost coefficient***. Note that the reduced cost coefficients corresponding to basic variables are zero.
- Theorem 16.2: A basic feasible solution is optimal if and only if the corresponding reduced cost coefficients are all nonnegative.

The Simplex Algorithm

- 1. Form a canonical augmented matrix corresponding to an initial basic feasible solution
- 2. Calculate the reduced cost coefficients corresponding to the nonbasic variables
- 3. If $r_j \geq 0$ for all j, stop – the current basic feasible solution is optimal.
- 4. Select a q such that $r_q < 0$
- 5. If no $y_{iq} > 0$, stop – the problem is unbounded; else, calculate $p = \arg\min_i \{y_{i0}/y_{iq} : y_{iq} > 0\}$. (If more than one index i minimizes y_{i0}/y_{iq}, we let p be the smallest such index.
- 6. Update the canonical augmented matrix by pivoting about the (p, q)th element.
- 7. Go to step 2.

The Simplex Algorithm

- Theorem 16.3: Suppose that we have an LP problem in standard form that has an optimal feasible solution. If the simplex method applied to this problem terminates and the reduced cost coefficients in the last step are all nonnegative, then the resulting basic feasible solution is optimal.

Example

- Consider the following linear program

$$\begin{aligned} &\text{maximize } 2x_1 + 5x_2 \\ &\text{subject to } x_1 \le 4 \\ &\qquad\qquad\quad x_2 \le 6 \\ &\qquad\qquad\quad x_1 + x_2 \le 8 \\ &\qquad\qquad\quad x_1, x_2 \ge 0 \end{aligned}$$

- Introducing slack variables, we transform the problem into standard form:

$$\begin{aligned} &\text{minimize } -2x_1 - 5x_2 \\ &\text{subject to } x_1 + x_3 = 4 \\ &\qquad\qquad\quad x_2 + x_4 = 6 \\ &\qquad\qquad\quad x_1 + x_2 + x_5 = 8 \\ &\qquad\qquad\quad x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

- The starting canonical augmented matrix for this problem is

$$\begin{array}{cccccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{b} \\ 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ 1 & 1 & 0 & 0 & 1 & 8 \end{array}$$

$$\begin{array}{cccccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{b} \\ 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ 1 & 1 & 0 & 0 & 1 & 8 \end{array}$$

Example

- Observe that the columns forming the identity matrix in the canonical augmented matrix above do not appear at the beginning. We could rearrange the augmented matrix so that the identity matrix would appear first. However, this is not essential from the computational point of view.
- The starting basic feasible solution to the problem in standard form is $\boldsymbol{x} = [0, 0, 4, 6, 8]^T$. The columns $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ are basic, and they form the identity matrix. The basis matrix is $\boldsymbol{B} = [\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5] = \boldsymbol{I}_3$
- The value of the objective function corresponding to this basic feasible solution is $z = 0$. We next compute the reduced cost coefficients corresponding to the nonbasic variables x_1, x_2

$$r_1 = c_1 - z_1 = c_1 - (c_3 y_{11} + c_4 y_{21} + c_5 y_{31}) = -2$$

$$r_2 = c_2 - z_2 = c_2 - (c_3 y_{12} + c_4 y_{22} + c_5 y_{32}) = -5$$

Example

- We would like now to move to an adjacent basic feasible solution for which the objective function value is lower. Naturally, if there is more than one such solution, it is desirable to move to the adjacent basic feasible solution with the lowest objective value. A common practice is to select the most negative value of r_j and then to bring the corresponding column into the basis.
- In this example, we bring $\boldsymbol{a}_2$ into the basis; that is, we choose $\boldsymbol{a}_2$ as the new basic column. We then compute $p = \arg\min\{y_{i0}/y_{i2} : y_{i2} > 0\} = 2$. We now update the canonical augmented matrix by pivoting about the (2,2)th entry using the pivot equations:

$$y'_{ij} = y_{ij} - \frac{y_{2j}}{y_{22}} y_{i2}, i \neq 2 \qquad y'_{2j} = \frac{y_{2j}}{y_{22}}$$

Example

- The resulting updated canonical augmented matrix is

$$\begin{array}{cccccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{b} \\ 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & -1 & 1 & 2 \end{array}$$

  Note that $\boldsymbol{a}_2$ entered the basis and $\boldsymbol{a}_4$ left the basis. The corresponding basic feasible solution is $\boldsymbol{x} = [0, 6, 4, 0, 2]^T$. We now compute the reduced cost coefficients for the nonbasic columns

$$r_1 = c_1 - z_1 = -2$$
$$r_4 = c_4 - z_4 = 5$$

  Because $r_1 = -2 < 0$, the current solution is not optimal, and a lower objective function value can be obtained by bringing $\boldsymbol{a}_1$ into the basis.

Example

- Proceeding to update the canonical augmented matrix by pivoting about the (3,1)th element, we obtain

$$\begin{array}{cccccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{b} \\ 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & -1 & 1 & 2 \end{array}$$

  The corresponding basic feasible solution is $\boldsymbol{x} = [2, 6, 2, 0, 0]^T$. The reduced cost coefficients are

$$r_4 = c_4 - z_4 = 3$$
$$r_5 = c_5 - z_5 = 2$$

  Because no reduced cost coefficient is negative, the current basic feasible solution is optimal. The solution to the original problem is therefore $x_1 = 2, x_2 = 6$, and the objective function value is 34.

Matrix Form of The Simplex Method

- Consider a linear programming problem in standard form

$$\begin{aligned} &\text{minimize } \boldsymbol{c}^T \boldsymbol{x} \\ &\text{subject to } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \quad \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Let the first m columns of $\boldsymbol{A}$ be the basic columns. The columns form a square $m \times m$ nonsingular matrix $\boldsymbol{B}$. The nonbasic columns of $\boldsymbol{A}$ form an $m \times (n-m)$ matrix $\boldsymbol{D}$. We partition the cost vector correspondingly as $\boldsymbol{c}^T = [\boldsymbol{c}_B^T, \boldsymbol{c}_D^T]$
Then, the original linear program can be represented as follows:

$$\text{minimize } \boldsymbol{c}_B^T \boldsymbol{x}_B + \boldsymbol{c}_D^T \boldsymbol{x}_D$$

$$\text{subject to } [\boldsymbol{B}, \boldsymbol{D}] \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_D \end{bmatrix} = \boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{D}\boldsymbol{x}_D = \boldsymbol{b}$$

$$\boldsymbol{x}_B \geq \boldsymbol{0}, \boldsymbol{x}_D \geq \boldsymbol{0}$$

$$\text{minimize } \boldsymbol{c}_B^T \boldsymbol{x}_B + \boldsymbol{c}_D^T \boldsymbol{x}_D$$
$$\text{subject to } [\boldsymbol{B}, \boldsymbol{D}] \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_D \end{bmatrix} = \boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{D}\boldsymbol{x}_D = \boldsymbol{b}$$

Matrix Form of The Simplex Method

- If $\boldsymbol{x}_D = \boldsymbol{0}$, then the solution $\boldsymbol{x} = [\boldsymbol{x}_B^T, \boldsymbol{x}_D^T]^T = [\boldsymbol{x}_B^T, \boldsymbol{0}^T]^T$ is the basic feasible solution corresponding to the basis $\boldsymbol{B}$. It is clear that for this to be a solution, we need $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$; that is, the basic feasible solution is

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$$

- The corresponding objective function value is $z_0 = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b}$
- If, on the other hand, $\boldsymbol{x}_D \neq \boldsymbol{0}$, then the solution $\boldsymbol{x} = [\boldsymbol{x}_B^T, \boldsymbol{x}_D^T]^T$ is not basic. In this case, $\boldsymbol{x}_B$ is given by $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{D}\boldsymbol{x}_D$ and the corresponding objective function value is

$$\begin{aligned} z &= \boldsymbol{c}_B^T \boldsymbol{x}_B + \boldsymbol{c}_D^T \boldsymbol{x}_D \\ &= \boldsymbol{c}_B^T(\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{D}\boldsymbol{x}_D) + \boldsymbol{c}_D^T \boldsymbol{x}_D \\ &= \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} + (\boldsymbol{c}_D^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{D})\boldsymbol{x}_D \\ &= z_0 + \boldsymbol{r}_D^T \boldsymbol{x}_D \end{aligned}$$

$$\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{D}$$

$$\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{D}$$

Matrix Form of The Simplex Method

- The elements of the vector $\boldsymbol{r}_D$ are the reduced cost coefficients corresponding to the nonbasic variables.
- If $\boldsymbol{r}_D \geq \boldsymbol{0}$, then the basic feasible solution corresponding to the basis $\boldsymbol{B}$ is optimal. If, on the other hand, a component of $\boldsymbol{r}_D$ is negative, then the value of the objective function can be reduced by increasing a corresponding components of $\boldsymbol{x}_D$ that is, by changing the basis.

Matrix Form of The Simplex Method

- We now use the foregoing observations to develop a matrix forms of the simplex method. To this end we first add the cost coefficient vector $\boldsymbol{c}^T$ to the bottom of the augmented matrix $[\boldsymbol{A}, \boldsymbol{b}]$

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c}^T & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{b} \\ \boldsymbol{c}_B^T & \boldsymbol{c}_D^T & 0 \end{bmatrix}$$

  We refer to this matrix as the ***tableau*** of the given LP problem. The tableau contains all relevant information about the linear program.

- Suppose that we apply elementary row operations to the tableau such that the top part of the tableau corresponding to the augmented matrix $[\boldsymbol{A}, \boldsymbol{b}]$ is transformed into canonical form. This corresponds to premultiplying the tableau by the matrix

$$\begin{bmatrix} \boldsymbol{B}^{-1} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

Matrix Form of The Simplex Method

- The result of this operation is

$$\begin{bmatrix} \boldsymbol{B}^{-1} & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{b} \\ \boldsymbol{c}_B^T & \boldsymbol{c}_D^T & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{c}_B^T & \boldsymbol{c}_D^T & 0 \end{bmatrix}$$

- We now apply elementary row operations to the tableau above so that the entries of the last row corresponding to the basic columns become zero. Specifically, this corresponds to premultiplication of the tableau by the matrix

$$\begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{0} \\ -\boldsymbol{c}_B^T & 1 \end{bmatrix}$$

The result is

$$\begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{0} \\ -\boldsymbol{c}_B^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{c}_B^T & \boldsymbol{c}_D^T & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0}^T & \boldsymbol{c}_D^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{D} & -\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} \end{bmatrix}$$

Matrix Form of The Simplex Method

- We refer to the resulting tableau as the ***canonical tableau corresponding to the basis*** $\boldsymbol{B}$. Note that the first m entries of the last column of the canonical tableau, $\boldsymbol{B}^{-1}\boldsymbol{b}$, are the values of the basic variables corresponding to the basis $\boldsymbol{B}$. The entries $\boldsymbol{c}_D^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{D}$ in the last row are the reduced cost coefficients. The last element in the last row of the tableau, $-\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b}$, is the negative of the value of the objective function corresponding to the basic feasible solution.

$$\begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{0} \\ -\boldsymbol{c}_B^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{c}_B^T & \boldsymbol{c}_D^T & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0}^T & \boldsymbol{c}_D^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{D} & -\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} \end{bmatrix}$$

Matrix Form of The Simplex Method

- Given an LP problem, we can in general construct many different canonical tableaus, depending on which columns are basic. Suppose that we have a canonical tableau corresponding to the particular basis. Consider the task of computing the tableau corresponding to another basis that differs from the previous basis by a single vector. This can be accomplished by applying elementary row operations to the tableau in a similar fashion as discussed above. We refer to this operation as ***updating*** the canonical tableau.

Matrix Form of The Simplex Method

- Note that updating of the tableau involves using exactly the same update equations as we used before in updating the canonical augmented matrix, namely, for $i = 1, ..., m+1$

$$y'_{ij} = y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq}, i \neq p$$

$$y'_{pj} = \frac{y_{pj}}{y_{pq}}$$

where y_{ij} and y'_{ij} are the (i, j)th entries of the original and updated canonical tableaus, respectively.

- Working with the tableau is a convenient way of implementing the simplex algorithm, since updating the tableau immediately gives us the values of both the basic variables and the reduce cost coefficients. In addition, the value of the objective function can be found in the lower right-hand corner of the tableau.

Example

- Consider the following linear programming problem

$$\begin{aligned} \text{maximize } & 7x_1 + 6x_2 \\ \text{subject to } & 2x_1 + x_2 \le 3 \\ & x_1 + 4x_2 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

- Transform the problem into standard form. Multiplying the objective function by -1, and introducing two nonnegative slack variables x_3, x_4, and construct the tableau for the problem

$$\begin{array}{cccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{b} \\ & 2 & 1 & 1 & 0 & 3 \\ & 1 & 4 & 0 & 1 & 4 \\ \boldsymbol{c}^T & -7 & -6 & 0 & 0 & 0 \end{array}$$

Example

$$\begin{array}{cccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{b} \\ & 2 & 1 & 1 & 0 & 3 \\ & 1 & 4 & 0 & 1 & 4 \\ \boldsymbol{c}^T & -7 & -6 & 0 & 0 & 0 \end{array}$$

- Notice that this tableau is already in canonical form with respect to the basis $[\boldsymbol{a}_3, \boldsymbol{a}_4]$. Hence, the last row contains the reduced cost coefficients, and the rightmost column contains the values of the basic variables. Because $\boldsymbol{r}_1 = -7$ is the most negative reduced cost coefficient, we bring $\boldsymbol{a}_1$ into the basis. We then compute the ratios $y_{10}/y_{11} = 3/2$ and $y_{20}/y_{21} = 4$. Because $y_{10}/y_{11} < y_{20}/y_{21}$, we get $p = \arg\min_i \{y_{i0}/y_{i1} : y_{i1} > 0\} = 1$ We pivot about the (1,1)th element of the tableau to obtain

$$\begin{array}{ccccc} 1 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & \frac{7}{2} & -\frac{1}{2} & 1 & \frac{5}{2} \\ 0 & -\frac{5}{2} & \frac{7}{2} & 0 & \frac{21}{2} \end{array}$$

$$\begin{array}{ccccc} 1 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & \frac{7}{2} & -\frac{1}{2} & 1 & \frac{5}{2} \\ 0 & -\frac{5}{2} & \frac{7}{2} & 0 & \frac{21}{2} \end{array}$$

Example

- In the second tableau, only r_2 is negative. Therefore, $q = 2$ (i.e., we bring $\boldsymbol{a}_2$ into the basis). Because

$$\frac{y_{10}}{y_{12}} = 3 \qquad \frac{y_{20}}{y_{22}} = \frac{5}{7}$$

we have $p = 2$. We thus pivot about the (2,2)th element of the second tableau to obtain the third tableau

$$\begin{array}{ccccc} 1 & 0 & \frac{4}{7} & -\frac{1}{7} & \frac{8}{7} \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & \frac{5}{7} \\ 0 & 0 & \frac{22}{7} & \frac{5}{7} & \frac{86}{7} \end{array}$$

- Because the last row of the third tableau has no negative elements, we conclude that the basic feasible solution corresponding to the third tableau is optimal. Thus, $x_1 = 8/7$ $x_2 = 5/7$, $x_3 = 0$, $x_4 = 0$ is the solution, and the objective value is -86/7. The solution to the original problem is $x_1 = 8/7$ $x_2 = 5/7$ and the corresponding objective value is 86/7

Remark

- Degenerate basic feasible solutions may arise in the course of applying the simplex algorithm. In such as situation, the minimum ratio y_{i0}/y_{iq} is 0. Therefore, even though the basis changes after we pivot about the (p, q)th element, the basic feasible solution does not (and remains degenerate)
- It is possible that if we start with a basis corresponding to a degenerate solution, several iterations of the simplex algorithm will involve the same degenerate solution, and eventually the original basis will occur. The entire process will then repeat indefinitely, leading to what is called ***cycling***.

Remark

- Such a scenario, although rare in practice, is clearly undesirable. Fortunately, there is a simple rule for choosing q and p that eliminates the cycling problem

$$q = \min\{i : r_i < 0\}$$

$$p = \min\{j : y_{j0}/y_{jq} = \min_i\{y_{i0}/y_{iq} : y_{iq} > 0\}\}$$

Two-Phase Simplex Method

- The simplex method requires starting with a tableau for the problem in canonical form; that is, we need an initial basic feasible solution. A brute-force approach to finding a starting basic feasible solution is to choose m basic columns arbitrarily and transform the tableau for the problem into canonical form. If the rightmost column is positive, then we have a legitimate (initial) basic feasible solution. Otherwise, we would have to pick another candidate basis. Potentially, this brute-force procedure requires $\binom{n}{m}$ tries, and is therefore not practical.

Two-Phase Simplex Method

- Certain LP problems have obvious initial basic feasible solutions. For example, if we have constraints of the form $\boldsymbol{Ax} \leq \boldsymbol{b}$ and we add m slack variables $z_1, \ldots, z_m$, then the constraints in standard form become

$$[\boldsymbol{A}, \boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix} = \boldsymbol{b} \qquad \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix} \geq \boldsymbol{0}$$

where $\boldsymbol{z} = [z_1, \ldots, z_m]^T$. The obvious initial basic feasible solution is $\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{b} \end{bmatrix}$

and the basic variables are the slack variables.

Two-Phase Simplex Method

- Suppose that we are given a linear program in standard form:

$$\text{minimize } \boldsymbol{c}^T\boldsymbol{x}$$
$$\text{subject to } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \qquad \boldsymbol{x} \geq \boldsymbol{0}$$

  In general, an initial basic feasible solution is not always apparent. We therefore need a systematic method for finding an initial basic feasible solution for general LP problems so that the simplex method can be initialized.

- For this purpose, suppose that we are given an LP problem in standard form. Consider the following associated ***artificial problem***:

$$\text{minimize } y_1 + y_2 + \cdots + y_m$$

$$\text{subject to } [\boldsymbol{A}, \boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b} \qquad \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \geq \boldsymbol{0}$$

$$\boldsymbol{y} = [y_1, y_2, \ldots, y_m]$$

Two-Phase Simplex Method

- We call $\boldsymbol{y}$ the vector of ***artificial variables***. Note that the artificial problem has an obvious initial basic feasible solution:

$$\begin{bmatrix} \mathbf{0} \\ \boldsymbol{b} \end{bmatrix}$$

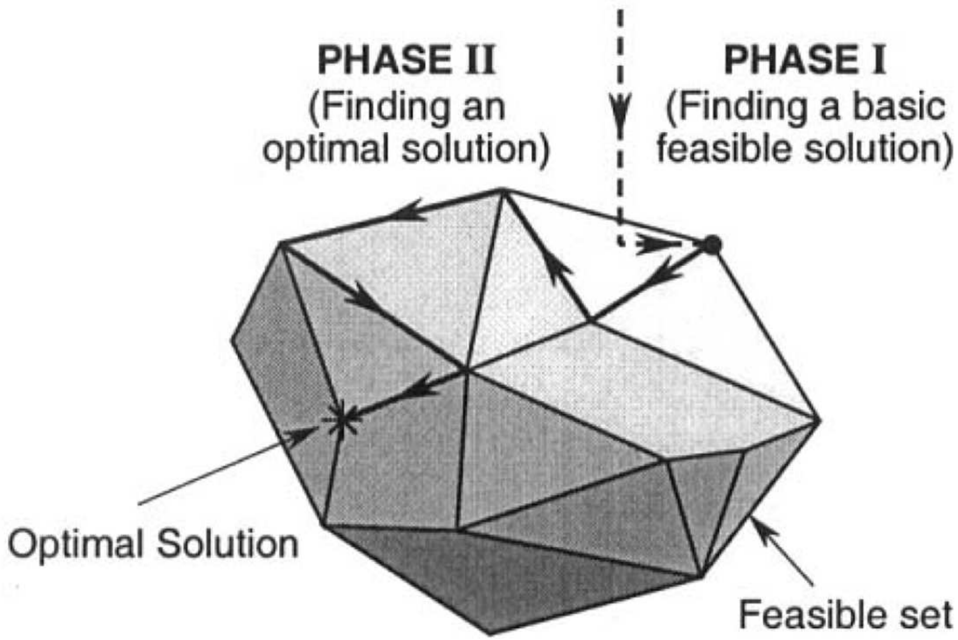
  We can therefore solve this problem by the simplex method.
- Proposition 16.1: The original LP problem has a basic feasible solution if and only if the associated artificial problem has an optimal feasible solution with objective function value zero.

Two-Phase Simplex Method

- Assume that the original LP problem has a basic feasible solution. Suppose that the simplex method applied to the associated artificial problem has terminated with an objective function value of zero. Then, the solution to the artificial problem will have all $y_i = 0, i = 1, ..., m$.
- Hence, assuming nondegeneracy, the basic variables are in the first n components; that is, none of the artificial variables are basic. Therefore, the first n components form a basic feasible solution to the original problem.
- We can then use this basic feasible solution as the initial basic feasible solution for the original LP problem (after deleting the components corresponding to artificial variables).

Two-Phase Simplex Method

- Thus, using artificial variables, we can attack a general linear programming problem by applying the ***two-phase simplex method***. In phase I we introduce artificial variables and the artificial objective function and find a basic feasible solution. In phase II we use the basic feasible solution resulting from phase I to initialize the simplex algorithm to solve the original LP problem.



Example

- Consider the following linear programming problem

$$\begin{aligned} \text{minimize } & 2x_1 + 3x_2 \\ \text{subject to } & 4x_1 + 2x_2 \geq 12 \\ & x_1 + 4x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

First, we express the problem in standard form by introducing surplus variables:

$$\begin{aligned} \text{minimize } & 2x_1 + 3x_2 \\ \text{subject to } & 4x_1 + 2x_2 - x_3 = 12 \\ & x_1 + 4x_2 - x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

There is no obvious basic feasible solution that we can use to initialize the simplex method. Therefore, we use the two-phase method.

Example

- Phase I. We introduce artificial variables $x_5, x_6 \geq 0$, and an artificial objective function $x_5 + x_6$. We form the corresponding tableau for the problem

$$\begin{array}{cccccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{a}_6 & \boldsymbol{b} \\ & 4 & 2 & -1 & 0 & 1 & 0 & 12 \\ & 1 & 4 & 0 & -1 & 0 & 1 & 6 \\ \boldsymbol{c}^T & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

  To initialize the simplex procedure, we must update the last row of this tableau to transform it into canonical form. We obtain

$$\begin{array}{ccccccc} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{a}_6 & \boldsymbol{b} \\ 4 & 2 & -1 & 0 & 1 & 0 & 12 \\ 1 & 4 & 0 & -1 & 0 & 1 & 6 \\ -5 & -6 & 1 & 1 & 0 & 0 & -18 \end{array}$$

Example

- The basic feasible solution corresponding to this tableau is not optimal. Therefore, we proceed with the simplex method to obtain the next tableau:

$$\begin{matrix} \frac{7}{2} & 0 & -1 & \frac{1}{2} & 1 & -\frac{1}{2} & 9 \\ \frac{1}{4} & 1 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{3}{2} \\ -\frac{7}{2} & 0 & 1 & -\frac{1}{2} & 0 & \frac{3}{2} & -9 \end{matrix}$$

- We still have not yet reached an optimal basic feasible solution. Performing another iteration, we get

$$\begin{matrix} 1 & 0 & -\frac{2}{7} & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \frac{18}{7} \\ 0 & 1 & \frac{1}{14} & -\frac{2}{7} & -\frac{1}{14} & \frac{2}{7} & \frac{6}{7} \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{matrix}$$

Both of the artificial variables have been driven out of the basis, and the current basic feasible solution is optimal.

Example

- Phase II. We start by deleting the columns corresponding to the artificial variables in the last tableau in phase I and revert back to the original objective function. We obtain

$$\begin{array}{cccccc} & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{b} \\ & 1 & 0 & -\frac{2}{7} & \frac{1}{7} & \frac{18}{7} \\ & 0 & 1 & \frac{1}{14} & -\frac{2}{7} & \frac{6}{7} \\ \boldsymbol{c}^T & 2 & 3 & 0 & 0 & 0 \end{array}$$

- We transform the last row so that the zeros appear in the basis columns; that is, we transform the tableau above into canonical form

$$\begin{array}{cccccc} 1 & 0 & -\frac{2}{7} & \frac{1}{7} & \frac{18}{7} \\ 0 & 1 & \frac{1}{14} & -\frac{2}{7} & \frac{6}{7} \\ 0 & 0 & \frac{5}{14} & \frac{4}{7} & -\frac{54}{7} \end{array}$$

All the reduced cost coefficients are nonnegative. Hence, the optimal solution is $\boldsymbol{x} = \left[\frac{18}{7}, \frac{6}{7}, 0, 0\right]^T$ and the optimal cost is 54/7.

Revised Simplex Method

- Consider an LP problem in standard form with a matrix $\boldsymbol{A}$ of size $m \times n$. Suppose that we use the simplex method to solve the problem. Experience suggests that if m is much smaller than n, then, in most instances, pivots will occur in only a small fraction of the columns of the matrix $\boldsymbol{A}$.
- The operation of pivoting involves updating all the columns of the tableau. However, if a particular column of $\boldsymbol{A}$ never enters any basis during the entire simplex procedure, then computations performed on this column are never used.
- Therefore, if m is much smaller than n, the effort expended on performing operations on many of the columns of $\boldsymbol{A}$ may be wasted. The ***revised simplex method*** reduces computation.

The Revised Simplex Method

- To be specific, suppose we are at a particular iteration in the simplex algorithm. Let $\boldsymbol{B}$ be the matrix composed of columns of $\boldsymbol{A}$ forming the current basis, and let $\boldsymbol{D}$ be the matrix composed of the remaining columns of $\boldsymbol{A}$.
- The sequence of elementary row operations on the tableau leading to this iteration (represented by matrices $\boldsymbol{E}_1, ..., \boldsymbol{E}_k$) corresponds to premultiplying $\boldsymbol{B}, \boldsymbol{D}, \boldsymbol{b}$ by $\boldsymbol{B}^{-1} = \boldsymbol{E}_k ... \boldsymbol{E}_1$.
- In particular, the vector of current values of basic variables is $\boldsymbol{B}^{-1}\boldsymbol{b}$. Observe that computation of the current basic feasible solution does not require computation of $\boldsymbol{B}^{-1}\boldsymbol{D}$. Instead, we only keep track of the basic variables and the revised tableau, which is the tableau $[\boldsymbol{B}^{-1}, \boldsymbol{B}^{-1}\boldsymbol{b}]$

$[\boldsymbol{B}^{-1}, \boldsymbol{B}^{-1}\boldsymbol{b}]$

The Revised Simplex Method

- Note that this tableau is only of size $m \times (m+1)$. To see how to update the revised tableau, suppose that we choose the column $\boldsymbol{a}_q$ to enter the basis. Let $\boldsymbol{y}_q = \boldsymbol{B}^{-1}\boldsymbol{a}_q$, $\boldsymbol{y}_0 = [y_{01}, \ldots, y_{0m}]^T = \boldsymbol{B}^{-1}\boldsymbol{b}$ and $p = \arg\min_i \{y_{i0}/y_{iq} : y_{iq} > 0\}$ (as the original simplex method). Then, to update the revised tableau, we form the augmented tableau $[\boldsymbol{B}^{-1}, \boldsymbol{y}_0, \boldsymbol{y}_q]$, and pivot about the p th element of the last column.
- We claim that the first $m+1$ columns of the resulting matrix comprise the revised tableau. To see this, write $\boldsymbol{B}^{-1} = \boldsymbol{E}_k \cdots \boldsymbol{E}_1$ and let the matrix $\boldsymbol{E}_{k+1}$ represent the pivoting operation above (i.e., $\boldsymbol{E}_{k+1}\boldsymbol{y}_q = \boldsymbol{e}_p$, the p th column of the $m \times m$ identity matrix)

The Revised Simplex Method

- The matrix $\boldsymbol{E}_{k+1}$ is given by

$$\boldsymbol{E}_{k+1} = \begin{bmatrix} 1 & & -y_{1q}/y_{pq} & & 0 \\ & \ddots & \vdots & & \\ & & 1/y_{pq} & & \\ & & \vdots & \ddots & \\ 0 & & -y_{mq}/y_{pq} & & 1 \end{bmatrix}$$

- Then, the updated augmented tableau resulting from the above pivoting operation is $[\boldsymbol{E}_{k+1}\boldsymbol{B}^{-1}, \boldsymbol{E}_{k+1}\boldsymbol{y}_0, \boldsymbol{e}_p]$. Let $\boldsymbol{B}_{new}$ be the new basis. Then, we have $\boldsymbol{B}_{new}^{-1} = \boldsymbol{E}_{k+1} \cdots \boldsymbol{E}_1$. But notice that $\boldsymbol{B}_{new}^{-1} = \boldsymbol{E}_{k+1}\boldsymbol{B}^{-1}$, and the values of the basic variables corresponding to $\boldsymbol{B}_{new}$ are given by $\boldsymbol{y}_{0new} = \boldsymbol{E}_{k+1}\boldsymbol{y}_0$. Hence, the updated tableau is indeed $[\boldsymbol{B}_{new}^{-1}, \boldsymbol{y}_{0new}] = [\boldsymbol{E}_{k+1}\boldsymbol{B}^{-1}, \boldsymbol{E}_{k+1}\boldsymbol{y}_0]$

The Revised Simplex Method

- 1. Form a revised tableau corresponding to an initial basic feasible solution $[\boldsymbol{B}^{-1}, \boldsymbol{y}_0]$
- 2. Calculate the current reduced cost coefficients vector via
  $\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \lambda^T \boldsymbol{D}$, where $\lambda^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$
- 3. If $r_j \geq 0$ for all j, stop – the current basic feasible solution is optimal.
- 4. Select a q such that $r_q < 0$ and compute $\boldsymbol{y}_q = \boldsymbol{B}^{-1}\boldsymbol{a}_q$
- 5. If no $y_{iq} > 0$, stop – the problem is unbounded; else, compute $p = \arg\min_i \{y_{i0}/y_{iq} : y_{iq} > 0\}$
- 6. Form the augmented revised tableau $[\boldsymbol{B}^{-1}, \boldsymbol{y}_0, \boldsymbol{y}_q]$, and pivot about the p th element of the last column. Form the updated revised tableau by taking the first $m+1$ columns of the resulting augmented revised tableau.
- 7. Go to step 2.

The Revised Simplex Method

- The reason for computing $\boldsymbol{r}_D$ in two steps indicated in Step 2 is as follows. We first note that $\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{D}$. To compute $\boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{D}$, we can either do the multiplication in the order $(\boldsymbol{c}_B^T \boldsymbol{B}^{-1})\boldsymbol{D}$ or $\boldsymbol{c}_B^T(\boldsymbol{B}^{-1}\boldsymbol{D})$. The former involves two vector-matrix multiplications, whereas the latter involves a matrix-matrix multiplication followed by a vector-matrix multiplication. Clearly the former is more efficient.
- As in the original simplex method, we can use the two-phase method to solve a given LP problem using the revised simple method. In particular, we use the revised tableau from the final step of phase I as the initial revised tableau in phase II.

Example

- Consider solving the following LP problem using the revised simplex method:

$$\begin{aligned} \text{maximize } & 3x_1 + 5x_2 \\ \text{subject to } & x_1 + x_2 \leq 4 \\ & 5x_1 + 3x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- First, we express the problem in standard form

$$\begin{aligned} \text{minimize } & -3x_1 - 5x_2 \\ \text{subject to } & x_1 + x_2 + x_3 = 4 \\ & 5x_1 + 3x_2 - x_4 = 8 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

There is no obvious basic feasible solution to this LP problem. Therefore, we use the two-phase method.

Example

- Phase I. We introduce one artificial variable x_5 and an artificial objective function. The tableau for the artificial problem is

$$
\begin{array}{ccccccc}
 & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{b} \\
 & 1 & 1 & 1 & 0 & 0 & 4 \\
 & 5 & 3 & 0 & -1 & 1 & 8 \\
\boldsymbol{c}^T & 0 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

We start with an initial basic feasible solution and corresponding $\boldsymbol{B}^{-1}$, as shown in the following revised tableau

$$
\begin{array}{c|cc|c}
 & \multicolumn{2}{c}{\boldsymbol{B}^{-1}} & \boldsymbol{y}_0 \\
\hline
x_3 & 1 & 0 & 4 \\
x_5 & 0 & 1 & 8
\end{array}
\qquad
\boldsymbol{c}_B^T = [0, 1]
$$

$$
\boldsymbol{D} = \begin{bmatrix} 1 & 1 & 0 \\ 5 & 3 & -1 \end{bmatrix}
$$

We compute $\boldsymbol{\lambda}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} = [0, 1]$

$$
\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \boldsymbol{\lambda}^T \boldsymbol{D} = [0, 0, 0] - [5, 3, -1] = [-5, -3, 1] = [r_1, r_2, r_4]
$$

Example

- Because r_1 is the most negative reduced cost coefficient, we bring $\boldsymbol{a}_1$ into the basis. To do this, we first compute $\boldsymbol{y}_1 = \boldsymbol{B}^{-1}\boldsymbol{a}_1$ In this case, $\boldsymbol{y}_1 = \boldsymbol{a}_1$. We get the augmented revised tableau

$$\begin{array}{c|cc|c|c} & \multicolumn{2}{c|}{\boldsymbol{B}^{-1}} & \boldsymbol{y}_0 & \boldsymbol{y}_1 \\ \hline x_3 & 1 & 0 & 4 & 1 \\ x_5 & 0 & 1 & 8 & 5 \end{array}$$

We then compute $p = \arg\min_i\{y_{i0}/y_{iq} : y_{iq} > 0\} = 2$ and pivot about the second element of the last column to get the updated revised tableau

$$\begin{array}{c|cc|c} & \multicolumn{2}{c|}{\boldsymbol{B}^{-1}} & \boldsymbol{y}_0 \\ \hline x_3 & 1 & -\frac{1}{5} & \frac{12}{5} \\ x_1 & 0 & \frac{1}{5} & \frac{8}{5} \end{array}$$

$$\boldsymbol{c}_B^T = [0, 0]$$

$$\boldsymbol{c}_D^T = [0, 0, 1]$$

We next compute $\boldsymbol{\lambda}^T = \boldsymbol{c}_B^T\boldsymbol{B}^{-1} = [0, 0]$

$$\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \boldsymbol{\lambda}^T\boldsymbol{D} = [0, 0, 1] = [r_2, r_4, r_5] \geq \boldsymbol{0}^T$$

All nonnegative. Hence, the solution to the artificial problem is $[8/5, 0, 12/5, 0, 0]^T$.

The initial basic feasible solution for phase II is therefore $[8/5, 0, 12/5, 0]^T$

$$
\begin{array}{c|cc|c}
 & \boldsymbol{B}^{-1} & & \boldsymbol{y}_0 \\
\hline
x_3 & 1 & -\frac{1}{5} & \frac{12}{5} \\
x_1 & 0 & \frac{1}{5} & \frac{8}{5}
\end{array}
$$

Example

- Phase II. The tableau for the original problem (in standard form) is

$$
\begin{array}{cccccc}
 & \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{b} \\
 & 1 & 1 & 1 & 0 & 4 \\
 & 5 & 3 & 0 & -1 & 8 \\
\boldsymbol{c}^T & -3 & -5 & 0 & 0 & 0
\end{array}
$$

As the initial revised tableau for phase II, we take the final revised tableau from phase I. We then compute

$$\boldsymbol{\lambda}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} = [0, -3] \begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & \frac{1}{5} \end{bmatrix} = [0, -\frac{3}{5}]$$

$$\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \boldsymbol{\lambda}^T \boldsymbol{D} = [-5, 0] - [0, -\frac{3}{5}] \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = [-\frac{16}{5}, -\frac{3}{5}] = [r_2, r_4]$$

$$
\begin{array}{c|ccc}
 & \boldsymbol{B}^{-1} & & \boldsymbol{y}_0 \\
\hline
x_3 & 1 & -\frac{1}{5} & \frac{12}{5} \\
x_1 & 0 & \frac{1}{5} & \frac{8}{5}
\end{array}
$$

Example

- We bring $\boldsymbol{a}_2$ into the basis, and compute $\boldsymbol{y}_2 = \boldsymbol{B}^{-1}\boldsymbol{a}_2$ to get

$$
\begin{array}{c|cccc}
 & \boldsymbol{B}^{-1} & & \boldsymbol{y}_0 & \boldsymbol{y}_2 \\
\hline
x_3 & 1 & -\frac{1}{5} & \frac{12}{5} & \frac{2}{5} \\
x_1 & 0 & \frac{1}{5} & \frac{8}{5} & \frac{3}{5}
\end{array}
$$

In this case, we get $p = 2$. We update this tableau by pivoting about the second element of the last column to get

$$
\begin{array}{c|ccc}
 & \boldsymbol{B}^{-1} & & \boldsymbol{y}_0 \\
\hline
x_3 & 1 & -\frac{1}{3} & \frac{4}{3} \\
x_2 & 0 & \frac{1}{3} & \frac{8}{3}
\end{array}
$$

We compute

$$
\boldsymbol{\lambda}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} = [0, -5] \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix} = [0, -\frac{5}{3}
$$

$$
\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \boldsymbol{\lambda}^T \boldsymbol{D} = [-3, 0] - [0, -\tfrac{5}{3}] \begin{bmatrix} 1 & 0 \\ 5 & -1 \end{bmatrix} = [\tfrac{16}{3}, -\tfrac{5}{3}] = [r_1, r_4]
$$

Example

- We now bring $\boldsymbol{a}_4$ into the basis

$$\begin{array}{c|cc|c|c} & \boldsymbol{B}^{-1} & & \boldsymbol{y}_0 & \boldsymbol{y}_4 \\ \hline x_3 & 1 & -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ x_2 & 0 & \frac{1}{3} & \frac{8}{3} & -\frac{1}{3} \end{array}$$

We update the tableau to obtain

$$\begin{array}{c|cc|c} & \boldsymbol{B}^{-1} & & \boldsymbol{y}_0 \\ \hline x_4 & 3 & -1 & 4 \\ x_2 & 1 & 0 & 4 \end{array}$$

We compute

$$\boldsymbol{\lambda}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1} = [0, -5] \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} = [-5, 0]$$

$$\boldsymbol{r}_D^T = \boldsymbol{c}_D^T - \boldsymbol{\lambda}^T \boldsymbol{D} = [-3, 0] - [-5, 0] \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix} = [2, 5] = [r_1, r_3]$$

The reduced cost coefficient are all positive. Hence, $[0, 4, 0, 4]^T$ is optimal. The optimal solution to the original problem is $[0,4]^T$