Chapter 6 Laplace Transforms

Advanced Engineering Mathematics

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Why Laplace Transforms?

- The process of solving an ODE using the Laplace transform method consists of three steps, shown schematically in Fig. 113:
  - ***Step 1.*** The given ODE is transformed into an algebraic equation, called the **subsidiary equation**.
  - ***Step 2.*** The subsidiary equation is solved by purely algebraic manipulations.
  - ***Step 3.*** The solution in Step 2 is transformed back, resulting in the solution of the given problem.

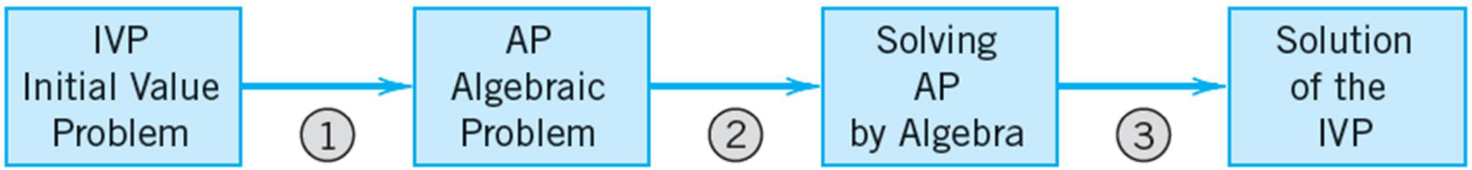

Fig. 113. Solving an IVP by Laplace transforms

Why Laplace Transforms?

- The key motivation for learning about Laplace transforms is that the process of solving an ODE is simplified to an algebraic problem (and transformations). This type of mathematics that converts problems of calculus to algebraic problems is known as operational calculus. The Laplace transform method has two main advantages over the methods discussed in Chaps. 1–4:

Why Laplace Transforms?

- I. Problems are solved more directly: Initial value problems are solved without first determining a general solution. Nonhomogenous ODEs are solved without first solving the corresponding homogeneous ODE.
- II. More importantly, the use of the unit step function (Heaviside function in Sec. 6.3) and Dirac's delta (in Sec. 6.4) make the method particularly powerful for problems with inputs (driving forces) that have discontinuities or represent short impulses or complicated periodic functions.

6.1 Laplace Transform. Linearity. First Shifting Theorem (*s*-Shifting)

Laplace Transform

- If $f(t)$ is a function defined for all $t \geq 0$, its **Laplace transform** is the integral of $f(t)$ times e^{-st} from $t = 0$ to ∞. It is a function of s, say, $F(s)$, and is denoted by $L(f)$; thus

(1) $$F(s) = L(f) = \int_0^\infty e^{-st} f(t) dt$$

- Here we must assume that $f(t)$ is such that the integral exists (that is, has some finite value). This assumption is usually satisfied in applications—we shall discuss this near the end of the section.

Laplace Transform

- Not only is the result $F(s)$ called the Laplace transform, but the operation just described, which yields $F(s)$ from a given $f(t)$, is also called the **Laplace transform**. It is an "**integral transform**"

$$F(s) = \int_0^\infty k(s,t) f(t) dt$$

with "**kernel**" $k(s, t) = e^{-st}$.

- Note that the Laplace transform is called an integral transform because it transforms (changes) a function in one space to a function in another space by a *process of integration* that involves a kernel. The kernel or kernel function is a function of the variables in the two spaces and defines the integral transform.

Laplace Transform

- Furthermore, the given function $f(t)$ in (1) is called the **inverse transform** of $F(s)$ and is denoted by $L^{-1}(f)$; that is, we shall write

(1*) $$f(t) = L^{-1}(F).$$

- Note that (1) and (1*) together imply $L^{-1}(L(f)) = f$ and $L(L^{-1}(F)) = F$.

Notation

- Original functions depend on t and their transforms on s—keep this in mind! Original functions are denoted by ***lowercase letters*** and their transforms by the same ***letters in capital***, so that $F(s)$ denotes the transform of $f(t)$, and $Y(s)$ denotes the transform of $y(t)$, and so on.

Example

- Let $f(t) = 1$ when $t \geq 0$. Find $F(s)$
- Solution. From (1) we obtain by integration

$$L(f) = L(1) = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$$

Such an integral is called an **improper integral** and, by definition, is evaluated according to the rule

$$\int_0^\infty e^{-st} f(t) dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t) dt$$

Hence our convenient notation means

$$\int_0^\infty e^{-st} dt = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-st} \right]_0^T = \lim_{T\to\infty} \left[-\frac{1}{s} e^{-sT} + \frac{1}{s} e^0 \right] = \frac{1}{s}$$

We shall use this notation throughout this chapter.

Example

- Let $f(t) = e^{at}$ when $t \geq 0$, where *a* is a constant. Find *L*(*f*).
- Solution. Again by (1),

$$L(e^{at}) = \int_0^\infty e^{-st} e^{at} dt = \frac{1}{a-s} e^{-(s-a)t} \Big|_0^\infty$$

hence, when *s*-*a* > 0,

$$L(e^{at}) = \frac{1}{s-a}$$

Theorem 1

- **Linearity of the Laplace Transform**
- *The Laplace transform is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose transforms exist and any constants a and b the transform of $af(t) + bg(t)$ exists, and*

$$L\{af(t)+bg(t)\} = aL\{f(t)\} + bL\{g(t)\}.$$

Proof of Theorem 1

- This is true because integration is a linear operation so that (1) gives

$$L\{af(t)+bg(t)\} = \int_0^\infty e^{-st}[af(t)+bg(t)]dt$$
$$= a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt$$
$$= aL\{f(t)\} + bL\{g(t)\}$$

Example

- Find the transforms of $\cosh at$ and $\sinh at$
- Solution. Since $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$ and $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$, we obtain from the previous example and Theorem 1

$$L(\cosh at) = \frac{1}{2}(L(e^{at}) + L(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}$$

$$L(\sinh at) = \frac{1}{2}(L(e^{at}) - L(e^{-at})) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2}$$

Example

- Derive the formulas

$$L(\cos\omega t) = \frac{s}{s^2+\omega^2} \qquad L(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$$

- Solution. We write $L_c = L(\cos\omega t)$ and $L_s = L(\sin\omega t)$ Integrating by parts and noting that the integral-free parts give no contribution from the upper limit ∞, we obtain

$$L_c = \int_0^\infty e^{-st}\cos\omega t dt = \frac{e^{-st}}{-s}\cos\omega t\bigg|_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st}\sin\omega t dt = \frac{1}{s} - \frac{\omega}{s}L_s$$

$$L_s = \int_0^\infty e^{-st}\sin\omega t dt = \frac{e^{-st}}{-s}\sin\omega t\bigg|_0^\infty + \frac{\omega}{s}\int_0^\infty e^{-st}\cos\omega t dt = \frac{\omega}{s}L_c$$

$$u = \cos\omega t, u' = -\omega\sin\omega t$$
$$v' = e^{-st}, v = -\tfrac{1}{s}e^{-st}$$

$$\int uv'dt = uv - \int u'vdt$$

Example

- By substituting L_s into the formula for L_c on the right and then by substituting L_c into the formula for L_s on the right, we obtain

$$L_c = \frac{1}{s} - \frac{\omega}{s}\left(\frac{\omega}{s}L_c\right) \qquad L_c\left(1 + \frac{\omega^2}{s^2}\right) = \frac{1}{s} \qquad L_c = \frac{s}{s^2 + \omega^2}$$

$$L_s = \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s}L_s\right) \qquad L_s\left(1 + \frac{\omega^2}{s^2}\right) = \frac{\omega}{s^2} \qquad L_s = \frac{\omega}{s^2 + \omega^2}$$

Laplace Transform

Table 6.1 **Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$**

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s - a}{(s - a)^2 + \omega^2}$
6	e^{at}	$\dfrac{1}{s - a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s - a)^2 + \omega^2}$

Laplace Transform

- From basic transforms almost all the others can be obtained by the use of the general properties of the Laplace transform. Formulas 1-3 are special cases of formula 4.
- We make the induction hypothesis that it holds for any integer $n \geq 0$:

$$L(t^{n+1}) = \int_0^\infty e^{-st} t^{n+1} dt = -\frac{1}{s} e^{-st} t^{n+1} \Big|_0^\infty + \frac{n+1}{s} \int_0^\infty e^{-st} t^n dt$$

now the integral-free part is zero and the last part is $(n+1)/s$ times $L(t^n)$. From this and the induction hypothesis,

$$L(t^{n+1}) = \frac{n+1}{s} L(t^n) = \frac{n+1}{s} \cdot \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}$$

Laplace Transform

- $\Gamma(a+1)$ in formula 5 is the so-called *gamma function*. We get formula 5 from (1), setting *st*=*x*:

  $$L(t^a) = \int_0^\infty e^{-st} t^a dt = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^a \frac{dx}{s} = \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^a dx$$

  where *s*>0. The last integral is precisely that defining $\Gamma(a+1)$, so we have $\Gamma(a+1)/s^{a+1}$, as claimed.
- Note the formula 4 also follows from 5 because $\Gamma(n+1) = n!$ for integer $n \geq 0$.

s-Shifting

- The Laplace transform has the very useful property that, if we know the transform of $f(t)$, we can immediately get that of $e^{at}f(t)$, as in Theorem 2.

Theorem 2

- First Shifting Theorem, s-Shifting
- *If $f(t)$ has the transform $F(s)$ (where $s > k$ for some k), then $e^{at}f(t)$ has the transform $F(s - a)$ (where $s - a > k$). In formulas,*

$$L\{e^{at}f(t)\} = F(s - a)$$

or, if we take the inverse on both sides,

$$e^{at}f(t) = L^{-1}\{F(s - a)\}$$

Proof of Theorem 2

- We obtain $F(s\text{-}a)$ by replacing s with $s\text{-}a$ in the integral in (1), so that

$$F(s-a)=\int_0^\infty e^{-(s-a)t}f(t)dt=\int_0^\infty e^{-st}[e^{at}f(t)]dt=L\{e^{at}f(t)\}$$

  If $F(s)$ exists (i.e., is finite) for s greater than some k, then our first integral exists for $s\text{-}a>k$. Now take the inverse on both sides of this formula to obtain the second formula in the theorem.

- CAUTION! $-a$ in $F(s\text{-}a)$ but $+a$ in $e^{at}f(t)$

Example

- From the previous example and the first shifting theorem we immediately obtain formulas 11 and 12 in Table 6.1

$$L(e^{at}\cos\omega t) = \frac{s-a}{(s-a)^2+\omega^2} \qquad L(e^{at}\sin\omega t) = \frac{\omega}{(s-a)^2+\omega^2}$$

  For instance, use these formulas to find the inverse of the transform $L(f) = \frac{3s-137}{s^2+2s+401}$

- Solution. Applying the inverse transform, using its linearity, and completing the square, we obtain

$$f = L^{-1}\left\{\frac{3(s+1)-140}{(s+1)^2+400}\right\} = 3L^{-1}\left\{\frac{s+1}{(s+1)^2+20^2}\right\} - 7L^{-1}\left\{\frac{20}{(s+1)^2+20^2}\right\}$$

  we now see that the inverse of the right side is damped vibration $f(t) = e^{-t}(3\cos 20t - 7\sin 20t)$

Example

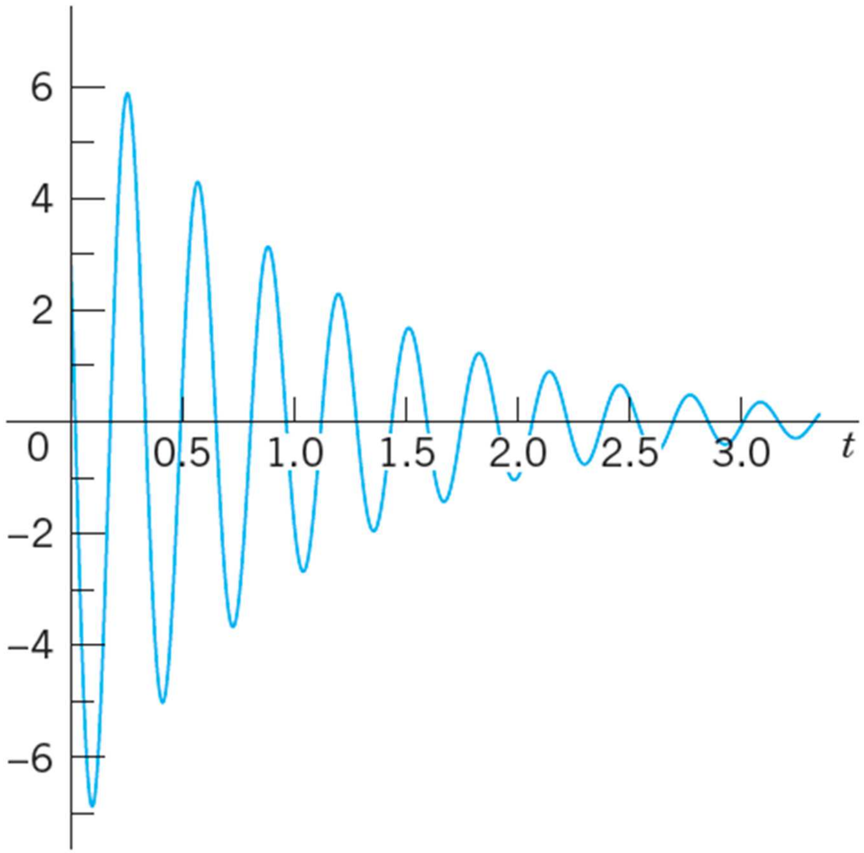

Fig. 114. Vibrations in Example 5

Existence and Uniqueness of Laplace Transforms

- A function $f(t)$ has a Laplace transform if it does not grow too fast, say, if for all $t \geq 0$ and some constants M and k it satisfies the "**growth restriction**"

(2) $$|f(t)| \leq Me^{kt}.$$

- $f(t)$ need not be continuous, but it should not be too bad. The technical term (generally used in mathematics) is *piecewise continuity*. $f(t)$ is **piecewise continuous** on a finite interval $a \leq t \leq b$ where f is defined, if this interval can be divided into *finitely many* subintervals in each of which f is continuous and has finite limits as t approaches either endpoint of such a subinterval from the interior.

Existence and Uniqueness of Laplace Transforms

- This then gives **finite jumps** as in Fig. 115 as the only possible discontinuities, but this suffices in most applications, and so does the following theorem.

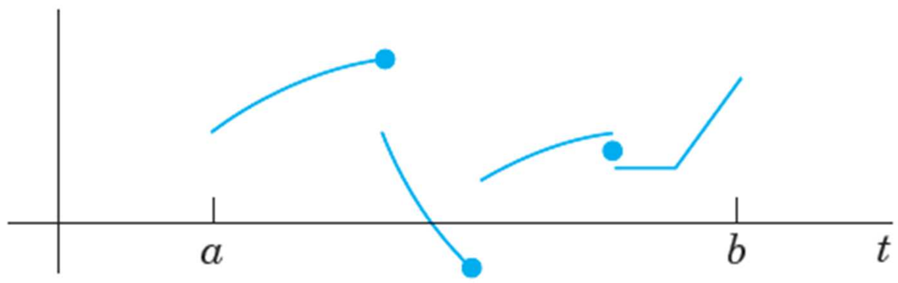

Fig. 115. Example of a piecewise continuous function $f(t)$. (The dots mark the function values at the jumps.)

Theorem 3

- **Existence Theorem for Laplace Transforms**
- *If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and satisfies* (2) *for all and some constants M and k, then the Laplace transform $L(f)$ exists for all $s > k$.*

Proof of Theorem 3

- Since *f*(*t*) is piecewise continuous, $e^{-st}f(t)$ is integrable over any finite interval on the *t*-axis. From (2), assuming that $s > k$ (to be needed for the existence of the last of the following integrals), we obtain the proof of *L*(*f*) from

$$|L(f)| = \left|\int_0^\infty e^{-st}f(t)dt\right| \le \int_0^\infty |f(t)|e^{-st}dt \le \int_0^\infty Me^{kt}e^{-st}dt = \frac{M}{s-k}$$

Uniqueness

- If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points. Hence, we may say that the inverse of a given transform is essentially unique. In particular, if two continuous functions have the same transform, they are completely identical.

6.2 Transforms of Derivatives and Integrals.

Laplace Transform

- The Laplace transform is a method of solving ODEs and initial value problems. The crucial idea is that ***operations of calculus on functions are replaced by operations of algebra on transforms***. Roughly, *differentiation* of $f(t)$ will correspond to *multiplication* of $L(f)$ by s (see Theorems 1 and 2) and *integration* of $f(t)$ to *division* of $L(f)$ by s.
- To solve ODEs, we must first consider the Laplace transform of derivatives. You have encountered such an idea in your study of logarithms. Under the application of the natural logarithm, a product of numbers becomes a sum of their logarithms, a division of numbers becomes their difference of logarithms.

Theorem 1

- **Laplace Transform of Derivatives**
- *The transforms of the first and second derivatives of $f(t)$ satisfy*

(1) $$L(f') = sL(f) - f(0)$$

(2) $$L(f'') = s^2L(f) - sf(0) - f'(0).$$

- *Formula* (1) *holds if $f(t)$ is continuous for all $t \geq 0$ and satisfies the growth restriction* (2) *in Sec.* 6.1 *and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geq 0$. Similarly,* (2) *holds if f and f' are continuous for all $t \geq 0$ and satisfy the growth restriction and f'' is piecewise continuous on every finite interval on the semi-axis $t \geq 0$.*

Proof of Theorem 1

- We prove (1) first under the additional assumption that f' is continuous. Then, by the definition and integration by parts,

$$L(f') = \int_0^\infty e^{-st} f'(t)dt = \left[e^{-st} f(t)\right]\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

Since f satisfies (2) in Sec. 6.1, the integrated part on the right is zero at the upper limit when $s > k$, and at the lower limit it contributes $-f(0)$. The last integral is $L(f)$. It exists for $s > k$ because of Theorem 3 in Sec. 6.1. Hence $L(f')$ exists when $s > k$ and (1) holds.

$$(1) \quad L(f') = sL(f) - f(0)$$

$$u = e^{-st}, u' = -se^{-st}$$
$$v' = f'(t), v = f(t)$$

$$\int uv'dt = uv - \int u'vdt$$

Proof of Theorem 1

- If *f*' is merely piecewise continuous, the proof is similar. In this case the interval of integration of *f*' must be broken up into parts such that *f*' is continuous in each such part.
- The proof of (2) now follows by applying (1) to *f*'' and then substituting (1), that is

$$L(f'') = sL(f') - f'(0) = s[sL(f) - f(0)] = s^2L(f) - sf(0) - f'(0)$$

Theorem 2

- **Laplace Transform of the Derivative $f^{(n)}$ of Any Order**
- *Let $f, f', \ldots, f^{(n-1)}$ be continuous for all $t \geq 0$ and satisfy the growth restriction* (2) *in Sec.* 6.1. *Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geq 0$. Then the transform of $f^{(n)}$ satisfies*

(3) $$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0).$$

Example 1

- Let $f(t) = t \sin \omega t$. Then $f(0) = 0, f'(t) = \sin \omega t + \omega t \cos \omega t$ $f'(0) = 0, f'' = 2\omega \cos \omega t - \omega^2 t \sin \omega t$. Hence by (2),

$$L(f'') = 2\omega \frac{s}{s^2 + \omega^2} - \omega^2 L(f) = s^2 L(f)$$

$$L(f) = L(t \sin \omega t) = \frac{2\omega s}{(s^2 + \omega^2)^2}$$

(2) $\quad L(f'') = s^2 L(f) - sf(0) - f'(0)$

Example 2

- This is a third derivation of $L(\cos \omega t)$ and $L(\sin \omega t)$.
  Let $f(t) = \cos \omega t$. Then
  $$f(0) = 1,\ f'(0) = 0,\ f''(t) = -\omega^2 \cos \omega t$$
  From this and (2) we obtain
  $$L(f'') = s^2 L(f) - s = -\omega^2 L(f)$$
  By algebra, $L(\cos \omega t) = \dfrac{s}{s^2 + \omega^2}$
- Similarly, let $g = \sin \omega t$. Then $g(0) = 0,\ g' = \omega \cos \omega t$
  From this and (1) we obtain
  $$L(g') = sL(g) = \omega L(\cos \omega t)$$
  Hence, $L(\sin \omega t) = \dfrac{\omega}{s} L(\cos \omega t) = \dfrac{\omega}{s^2 + \omega^2}$

(2) $\quad L(f'') = s^2 L(f) - sf(0) - f'(0)$

Laplace Transform of the Integral of a Function

- Differentiation and integration are inverse operations, and so are multiplication and division.
- Since differentiation of a function $f(t)$ (roughly) corresponds to multiplication of its transform $L(f)$ by s, we expect integration of $f(t)$ to correspond to division of $L(f)$ by s:

Theorem 3

- **Laplace Transform of Integral**
- *Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geq 0$ and satisfies a growth restriction (2), Sec. 6.1. Then, for $s > 0$, $s > k$, and $t > 0$,*

(4) $$L\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$ thus $$\int_0^t f(\tau)d\tau = L^{-1}\left\{\frac{1}{s}F(s)\right\}$$

Proof of Theorem 3

- Denote the integral in (4) by $g(t)$. Since $f(t)$ is piecewise continuous, $g(t)$ is continuous, and (2), Sec. 6.1, gives $(k > 0)$

$$|g(t)| = \left| \int_0^t f(\tau)d\tau \right| \le \int_0^t |f(\tau)|d\tau \le M \int_0^t e^{k\tau}d\tau = \frac{M}{k}(e^{kt} - 1) \le \frac{M}{k}e^{kt}$$

  This shows that $g(t)$ also satisfies a growth restriction. Also, $g'(t)=f(t)$, except at points at which $f(t)$ is discontinuous. Hence, $g'(t)$ is piecewise continuous on each finite interval and, by Theorem 1, since $g(0)=0$ (the integral from 0 to 0 is zero)

$$L\{f(t)\} = L\{g'(t)\} = sL\{g(t)\} - g(0) = sL\{g(t)\}$$

  Division by s and interchange of the left and right sides gives the first formula in (4), from which the second follows by taking the inverse transform on both sides.

Example 3

- Using Theorem 3, find the inverse of $\dfrac{1}{s(s^2+\omega^2)}$ and $\dfrac{1}{s^2(s^2+\omega^2)}$

- Solution. From Table 6.1 in Sec. 6.1 and the integration in (4) we obtain

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

$$L^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin \omega t}{\omega}$$

$$L^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \int_0^t \frac{\sin \omega \tau}{\omega} d\tau = \frac{1}{\omega^2}(1-\cos \omega t)$$

Example 3

- This is formula 19 in Sec. 6.9. Integrating this result again and using (4) as before, we obtain formula 20 in Sec. 6.9

$$L^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1-\cos\omega\tau)\,d\tau = \left[\frac{\tau}{\omega^2} - \frac{\sin\omega\tau}{\omega^3}\right] = \frac{t}{\omega^2} - \frac{\sin\omega t}{\omega^3}$$

It is typical that results such as these can be found in several ways. In this example, try partial fraction reduction.

Differential Equations, Initial Value Problems

- Let us now discuss how the Laplace transform method solves ODEs and initial value problems. We consider an initial value problem

(5) $y'' + ay' + by = r(t), \; y(0) = K_0, \quad y'(0) = K_1$

where *a* and *b* are constant. Here *r*(*t*) is the given **input** (*driving force*) applied to the mechanical or electrical system and *y*(*t*) is the **output** (*response to the input*) to be obtained.

Differential Equations, Initial Value Problems

- In Laplace's method we do three steps:
- ***Step 1. Setting up the subsidiary equation.*** This is an algebraic equation for the transform $Y = L(y)$ obtained by transforming (5) by means of (1) and (2), namely,

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where $R(s) = L(r)$. Collecting the Y-terms, we have the subsidiary equation

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

Differential Equations, Initial Value Problems

- ***Step 2. Solution of the subsidiary equation by algebra.*** We divide by $s^2 + as + b$ and use the so-called **transfer function**

(6) $$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2}$$

(Q is often denoted by H, but we need H much more frequently for other purposes.)

Differential Equations, Initial Value Problems

- This gives the solution

(7) $Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s).$

- If $y(0) = y'(0) = 0$, this is simply $Y = RQ$; hence

$$Q = \frac{Y}{R} = \frac{L(\text{output})}{L(\text{input})}$$

and this explains the name of Q. Note that ***Q depends neither on r(t) nor on the initial conditions*** (but only on a and b).

Differential Equations, Initial Value Problems

- ***Step 3. Inversion of Y to obtain y =*** $\mathcal{L}^{-1}$ **(*Y*).** We reduce (7) (usually by ***partial fractions*** as in calculus) to a sum of terms whose inverses can be found from the tables (e.g., in Sec. 6.1 or Sec. 6.9) or by a CAS, so that we obtain the solution $y(t) = \mathcal{L}^{-1}(Y)$ *of* (5).

Example 4

- Solve $y'' - y = t, \quad y(0) = 1, \quad y'(0) = 1$
- ***Solution. Step 1.*** From (2) and Table 6.1 we get the subsidiary equation [with $Y = L(y)$]

$$s^2Y - sy(0) - y'(0) - Y = 1/s^2,$$

thus $$(s^2 - 1)Y = s + 1 + 1/s^2.$$

- ***Step 2.*** The transfer function is $Q = 1/(s^2 - 1)$, and (7) becomes

$$Y = (s+1)Q + \frac{1}{s^2}Q = \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)}$$

- Simplification of the first fraction and an expansion of the last fraction gives

$$Y = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right)$$

Example 4

- ***Step 3.*** From this expression for Y and Table 6.1 we obtain the solution

$$y(t) = L^{-1}(Y) = L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{1}{s^2-1}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t$$

- The diagram summarizes this approach

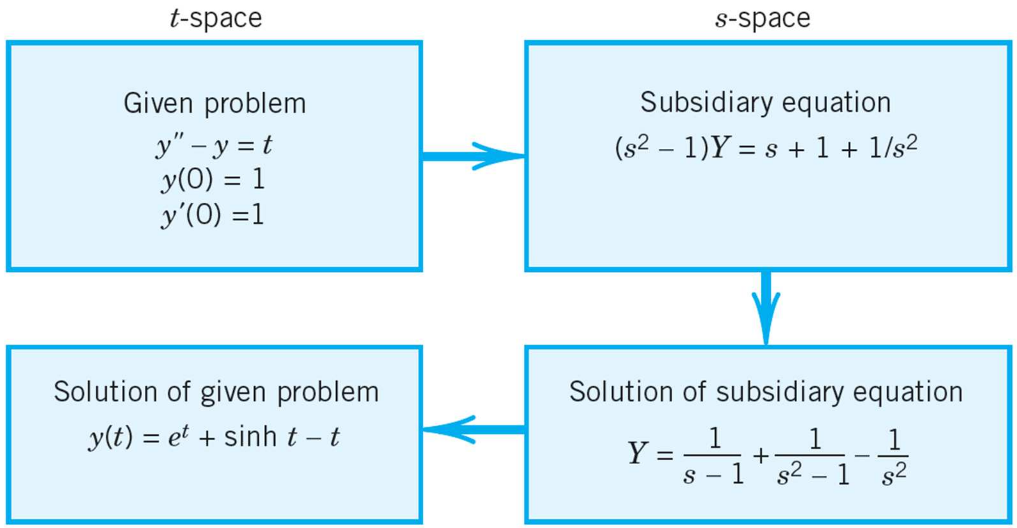

Fig. 116. Steps of the Laplace transform method

Example 5

- Solve the initial value problem

$$y'' + y' + 9y = 0. \quad y(0) = 0.16, \quad y'(0) = 0$$

- Solution. From (1) and (2) we see that the subsidiary equation is

$$s^2Y - 0.16s + sY - 0.16 + 9Y = 0$$

$$(s^2 + s + 9)Y = 0.16(s + 1)$$

$$\frac{0.16s + 0.08}{s^2 + \frac{35}{4}} = \frac{0.16s}{s^2 + \frac{35}{4}} + \frac{0.08}{s^2 + \frac{35}{4}}$$

The solution is

$$Y = \frac{0.16(s+1)}{s^2 + s + 9} = \frac{0.16(s + \frac{1}{2}) + 0.08}{(s + \frac{1}{2})^2 + \frac{35}{4}}$$

Hence by the first shifting theorem and the formulas for cos and sin in Table 6.1 we obtain

$$y(t) = L^{-1}(Y) = e^{-t/2}\left(0.16 \cos \sqrt{\frac{35}{4}}t + \frac{0.08}{\frac{1}{2}\sqrt{35}} \sin \sqrt{\frac{35}{4}}t\right)$$

$$= e^{-0.5t}(0.16 \cos 2.96t + 0.027 \sin 2.96t)$$

Table 6.1

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

$$(2) \qquad L(f'') = s^2 L(f) - s f(0) - f'(0)$$

Advantages of the Laplace Method

- **1.** *Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE*. See Example 4.
- **2.** *Initial values are automatically taken care of*. See Examples 4 and 5.
- **3.** *Complicated inputs* $r(t)$ (right sides of linear ODEs) *can be handled very efficiently*, as we show in the next sections.

Example 6

- This means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t=0$. For such a problem set $t = \tilde{t} + t_0$, so that $t = t_0$ gives $\tilde{t} = 0$ and the Laplace transform can be applied. For instance, solve

$$y'' + y = 2t, \quad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \quad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}$$

- Solution. We have $t_0 = \frac{1}{4}\pi$ and we set $t = \tilde{t} + \frac{1}{4}\pi$. Then the problem is

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{1}{4}\pi), \quad \tilde{y}(0) = \tfrac{1}{2}\pi, \quad \tilde{y}'(0) = 2 - \sqrt{2}$$

where $\tilde{y}(\tilde{t}) = y(t)$.

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{1}{4}\pi), \quad \tilde{y}(0) = \tfrac{1}{2}\pi, \quad \tilde{y}'(0) = 2 - \sqrt{2}$$

Example 6

- Using (2) and Table 6.1 and denoting the transform of $\tilde{y}$ by $\tilde{Y}$, we see that the subsidiary equation of the "shifted" initial value problem is

$$s^2\tilde{Y} - s \cdot \tfrac{1}{2}\pi - (2 - \sqrt{2}) + \tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s}$$

$$(s^2 + 1)\tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}$$

Solving this algebraically for $\tilde{Y}$, we obtain

$$\tilde{Y} = \frac{2}{(s^2 + 1)s^2} + \frac{\frac{1}{2}\pi}{(s^2 + 1)s} + \frac{\frac{1}{2}\pi s}{s^2 + 1} + \frac{2 - \sqrt{2}}{s^2 + 1}$$

The inverse of the first two terms can be seen from Example 3 (with $\omega = 1$), and the last two terms give cos and sin,

Example 6

$$\tilde{y} = L^{-1}(\tilde{Y}) = 2(\tilde{t} - \sin\tilde{t}) + \tfrac{1}{2}\pi(1 - \cos\tilde{t}) + \tfrac{1}{2}\pi\cos\tilde{t} + (2 - \sqrt{2})\sin\tilde{t}$$
$$= 2\tilde{t} + \tfrac{1}{2}\pi - \sqrt{2}\sin\tilde{t}$$

- Now $\tilde{t} = t - \tfrac{1}{4}\pi, \sin\tilde{t} = \dfrac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer is $y = 2t - \sin t + \cos t$

6.3 Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting)

Unit Step Function

- We shall introduce two auxiliary functions, the *unit step function* or *Heaviside function* $u(t - a)$ (following) and *Dirac's delta* $\delta(t - a)$ (in Sec. 6.4).
- These functions are suitable for solving ODEs with complicated right sides of considerable engineering interest, such as single waves, inputs (driving forces) that are discontinuous or act for some time only, periodic inputs more general than just cosine and sine, or impulsive forces acting for an instant (hammerblows, for example).

Unit Step Function

- The **unit step function** or **Heaviside function** $u(t-a)$ is 0 for $t < a$, has a jump of size 1 at $t = a$ (where we can leave it undefined), and is 1 for $t > a$, in a formula:

(1)
$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \qquad (a \geq 0).$$

- Figure 118 shows the special case $u(t)$, which has its jump at zero, and Fig. 119 the general case $u(t-a)$ for an arbitrary positive a. (For Heaviside, see Sec. 6.1.)

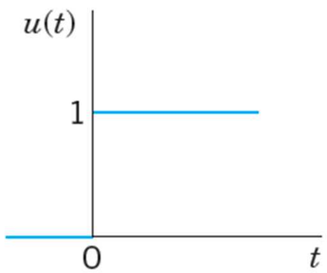

Fig. 118. Unit step function $u(t)$

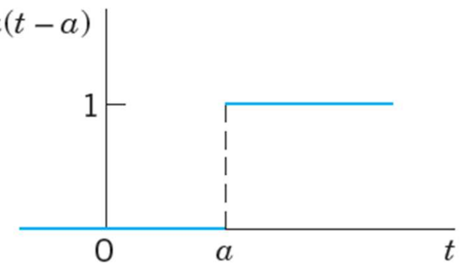

Fig. 119. Unit step function $u(t-a)$

Unit Step Function

- The transform of *u*(*t*-*a*) follows directly from the defining integral in Sec. 6.1,

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)dt = \int_0^\infty e^{-st} \cdot 1 dt = -\frac{e^{-st}}{s}\bigg|_{t=a}^{\infty}$$

  here the integration begins at *t* = *a* (≥0) because *u*(*t*-*a*) is 0 for *t* < *a*. Hence

$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

- The unit step function is a typical "engineering function" made to measure for engineering applications, which often involve functions that are either "off" or "on."

Unit Step Function

- Multiplying functions $f(t)$ with $u(t-a)$, we can produce all sorts of effects. The simple basic idea is illustrated in Figs. 120 and 121. In Fig. 120 the given function is shown in (A). In (B) it is switched off between $t = 0$ and $t = 2$ (because $u(t-2) = 0$ when $t < 2$) and is switched on beginning at $t = 2$. In (C) it is shifted to the right by 2 units, say, for instance, by 2 sec, so that it begins 2 sec later in the same fashion as before.

Unit Step Function

- More generally we have the following.
- *Let* $f(t) = 0$ *for all negative* t*. Then* $f(t - a)u(t - a)$ *with* $a > 0$ *is* $f(t)$ ***shifted*** (*translated*) *to the right by the amount* a.
- Figure 121 shows the effect of many unit step functions, three of them in (A) and infinitely many in (B) when continued periodically to the right; this is the effect of a rectifier that clips off the negative half-waves of a sinuosidal voltage.

Unit Step Function

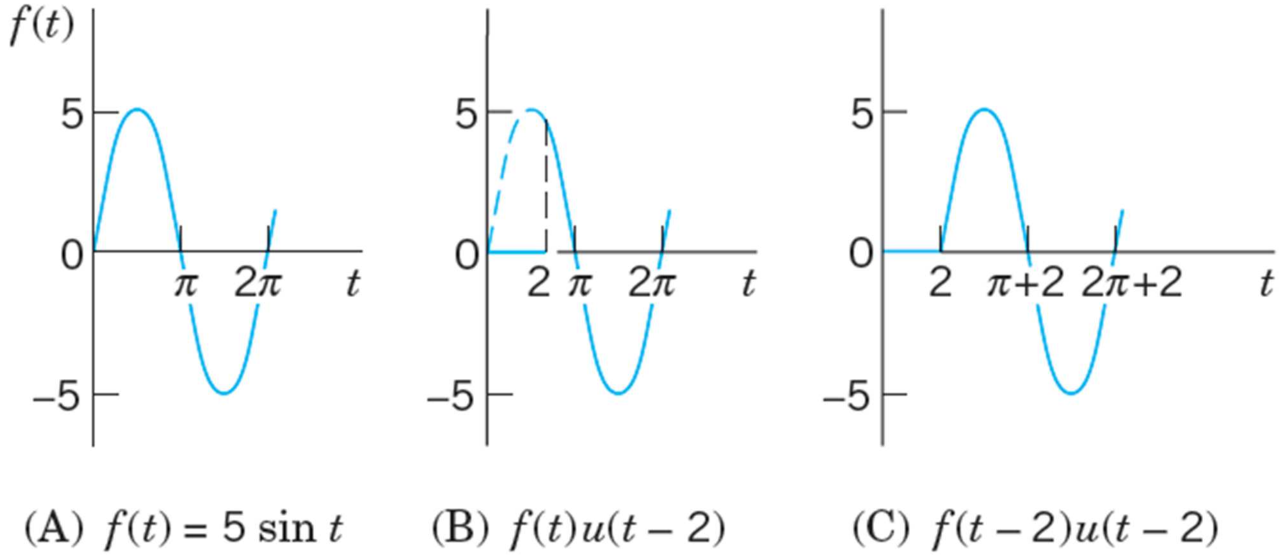

Fig. 120. Effects of the unit step function: (A) Given function. (B) Switching off and on. (C) Shift.

Unit Step Function

(A) $k[u(t-1) - 2u(t-4) + u(t-6)]$ (B) $4 \sin(\frac{1}{2}\pi t)[u(t) - u(t-2) + u(t-4) - + \cdots]$

Fig. 121. Use of many unit step functions.

Time Shifting (t-Shifting)

- The first shifting theorem ("s-shifting") in Sec. 6.1 concerned transforms $F(s) = L\{f(t)\}$ and $F(s - a) =$ $\mathrm{L}\{e^{at}f(t)\}$.
- The second shifting theorem will concern functions $f(t)$ and $f(t - a)$.

s-shifting: $L\{e^{at}f(t)\} = F(s-a)$

Time Shifting (*t*-Shifting)

- Theorem 1. Second Shifting Theorem; Time Shifting
- *If f(t) has the transform F(s) then the* "**shifted function**"

(3) $$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

has the transform $e^{-as}F(s)$. *That is, if* $L\{f(t)\} = F(s)$, *then*

(4) $$L\{f(t-a)u(t-a)\} = e^{-as}F(s).$$

Or, if we take the inverse on both sides, we can write

(4*) $$f(t-a)u(t-a)\} = L^{-1}\{e^{-as}F(s)\}.$$

Time Shifting (t-Shifting)

- Practically speaking, if we know $F(s)$, we can obtain the transform of (3) by multiplying $F(s)$ by e^{-as}. In Fig. 120, the transform of 5 $\sin t$ is $F(s) = 5/(s^2+1)$, hence the shifted function 5 $\sin(t-2)u(t-2)$ shown in Fig. 120 (C) has the transform

$$e^{-2s}F(s) = 5e^{-2s}/s^2 + 1$$

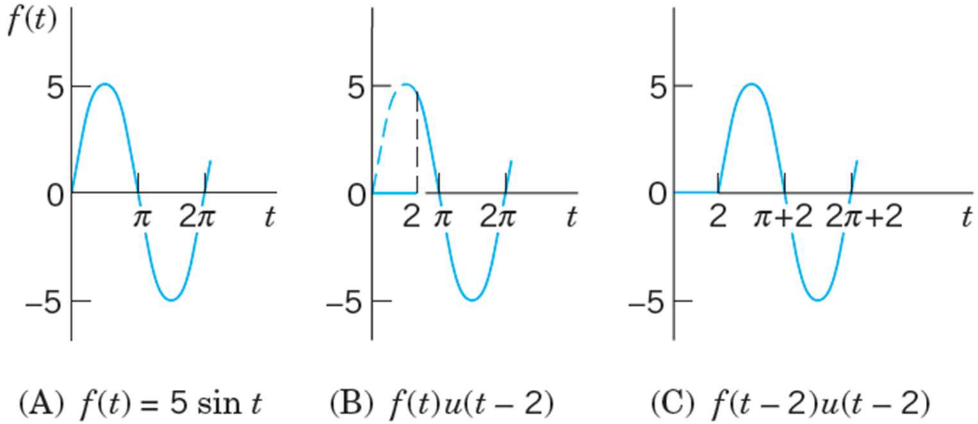

Fig. 120. Effects of the unit step function: (A) Given function. (B) Switching off and on. (C) Shift.

$$(4) \quad L\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

Time Shifting (t-Shifting)

- Proof of Theorem 1. In (4), on the right, we use the definition of the Laplace transform, writing τ for t (to have t available later). Then, taking e^{-as} inside the integral, we have

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau} f(\tau)d\tau = \int_0^\infty e^{-s(\tau+a)} f(\tau)d\tau$$

Substituting $\tau + a = t$, thus $\tau = t - a, d\tau = dt$ in the integral, we obtain

$$e^{-as}F(s) = \int_a^\infty e^{-st} f(t-a)dt$$

Time Shifting (*t*-Shifting)

- To make the right side into a Laplace transform, we must have an integral from 0 to ∞ , not from *a* to ∞ But this is easy. We multiply the integral by *u*(*t*-*a*). Then for *t* from 0 to *a* the integrand is 0, and we can write, with $\tilde{f}$ as in (3),

$$e^{-as}F(s) = \int_0^\infty e^{-st} f(t-a)u(t-a)dt = \int_0^\infty e^{-st}\tilde{f}(t)dt$$

  This integral is the left side of (4), the Laplace transform of $\tilde{f}(t)$ in (3). This completes the proof.

Example

- Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if } t > \frac{1}{2}\pi \end{cases}$$

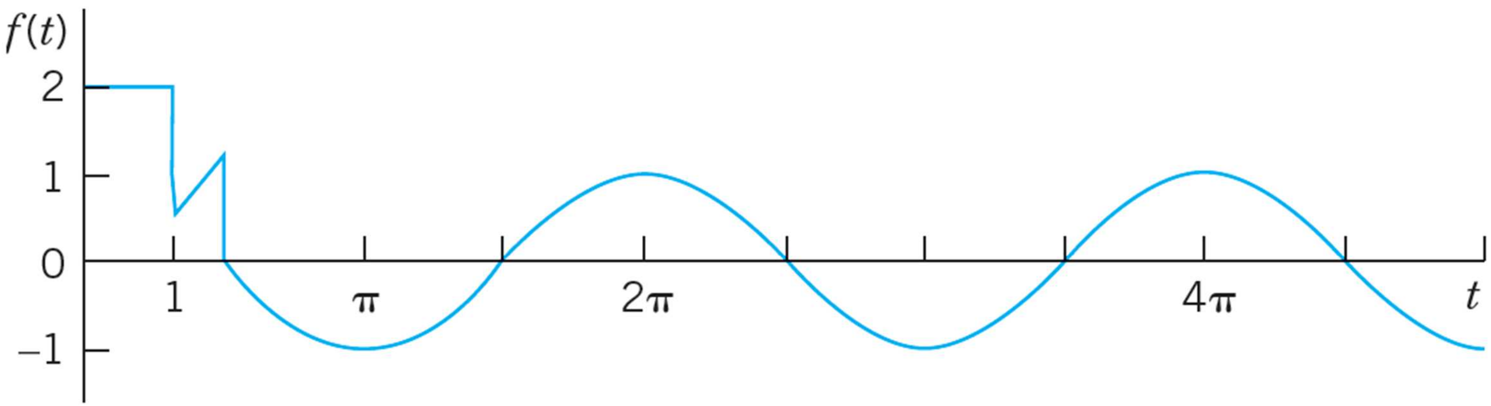

Fig. 122. $f(t)$ in Example 1

Table 6.1

Table 6.1 Some Functions $f(t)$ and Their Laplace Transforms $\mathcal{L}(f)$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a + 1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s - a}{(s - a)^2 + \omega^2}$
6	e^{at}	$\dfrac{1}{s - a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s - a)^2 + \omega^2}$

Example

- ***Step 1.*** In terms of unit step functions $f(t) = 2(1 - u(t - 1)) + \frac{1}{2}t^2(u(t - 1) - u(t - \frac{1}{2}\pi)) + (\cos t)u(t - \frac{1}{2}\pi)$. Indeed, $2(1 - u(t - 1))$ gives $f(t)$ for $0 < t < 1$, and so on.
- ***Step 2.*** To apply Theorem 1, we must write each term in $f(t)$ in the form $f(t - a)u(t - a)$. Thus, $2(1 - u(t - 1))$ remains as it is and gives the transform $2(1 - e^{-s})/s$. Then

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = \mathcal{L}\left\{\left(\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}\right)u(t-1)\right\} = \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s}$$

$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-\frac{1}{2}\pi)\right\} = \mathcal{L}\left\{\left(\frac{1}{2}(t-\frac{1}{2}\pi)^2 + \frac{\pi}{2}(t-\frac{1}{2}\pi) + \frac{\pi^2}{8}\right)u(t-\frac{1}{2}\pi)\right\}$$

$$= \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2}$$

Example

- ***Step 2.*** ***(continued)***

$$\mathcal{L}\left\{(\cos t)u(t-\frac{1}{2}\pi)\right\}=\mathcal{L}\left\{-\left(\sin(t-\frac{1}{2}\pi)\right)u(t-\frac{1}{2}\pi)\right\}=-\frac{1}{s^2+1}e^{-\pi s/2}.$$

- Together,

$$\mathcal{L}(f)=\frac{2}{s}-\frac{2}{s}e^{-s}+\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)e^{-s}-\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)e^{-\pi s/2}-\frac{1}{s^2+1}e^{-\pi s/2}.$$

- If the conversion of $f(t)$ to $f(t\text{-}a)$ is inconvenient, replace it by

(4**) $$L\{f(t)u(t-a)\}=e^{-as}L\{f(t+a)\}$$

Example

- Step 2. (continued). (4**) follows from (4) by writing *f*(*t*-*a*)=*g*(*t*), hence *f*(*t*)=*g*(*t*+*a*) and then again writing *f* for *g*. Thus,

$a = 1, f(t) = t^2$

$$L\left\{\frac{1}{2}t^2u(t-1)\right\} = e^{-s}L\left\{\frac{1}{2}(t+1)^2\right\} = e^{-s}L\left\{\frac{1}{2}t^2+t+\frac{1}{2}\right\} = e^{-s}\left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)$$

as before, Similarly for $L\{\frac{1}{2}t^2u(t-\frac{1}{2}\pi)\}$. Finally, by (4**),

$$L\left\{\cos t\, u\left(t-\frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}L\left\{\cos\left(t+\frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}L\{-\sin t\} = -e^{-\pi s/2}\frac{1}{s^2+1}$$

$a = \frac{1}{2}\pi, f(t) = \cos t$

(4) $\quad L\{f(t-a)u(t-a)\} = e^{-as}F(s) = e^{-as}L\{f(t)\}$

$$L\{g(t)u(t-a)\} = e^{-as}L\{g(t+a)\}$$

(4**) $\quad L\{f(t)u(t-a)\} = e^{-as}L\{f(t+a)\}$

s-shifting: $L\{e^{at}f(t)\} = F(s-a)$

Example

- Find the inverse transform *f*(*t*) of

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-2s}}{s^2+\pi^2} + \frac{e^{-3s}}{(s+2)^2}$$

- Solution. Without the exponential functions in the numerator the three terms of *F*(*s*) would have the inverses $(\sin \pi t)/\pi$, $(\sin \pi t)/\pi$, and te^{-2t} because $1/s^2$ has the inverse *t*, so that $1/(s+2)^2$ has the inverse te^{-2t} by the first shifting theorem in Sec. 6.1. Hence by the second shifting theorem (*t*-shifting),

$$f(t) = \frac{1}{\pi}\sin(\pi(t-1))u(t-1) + \frac{1}{\pi}\sin(\pi(t-2))u(t-2) + (t-3)e^{-2(t-3)}u(t-3)$$

Example

- Now $\sin(\pi t-\pi)=-\sin\pi t$ and $\sin(\pi t-2\pi)=\sin\pi t$, so that the first and second terms cancel each other when $t>2$. Hence we obtain $f(t)=0$ if $0<t<1$, $-\sin(\pi t)/\pi$ if $1<t<2$, 0 if $2<t<3$, and $(t-3)e^{-2(t-3)}$ if $t>3$.

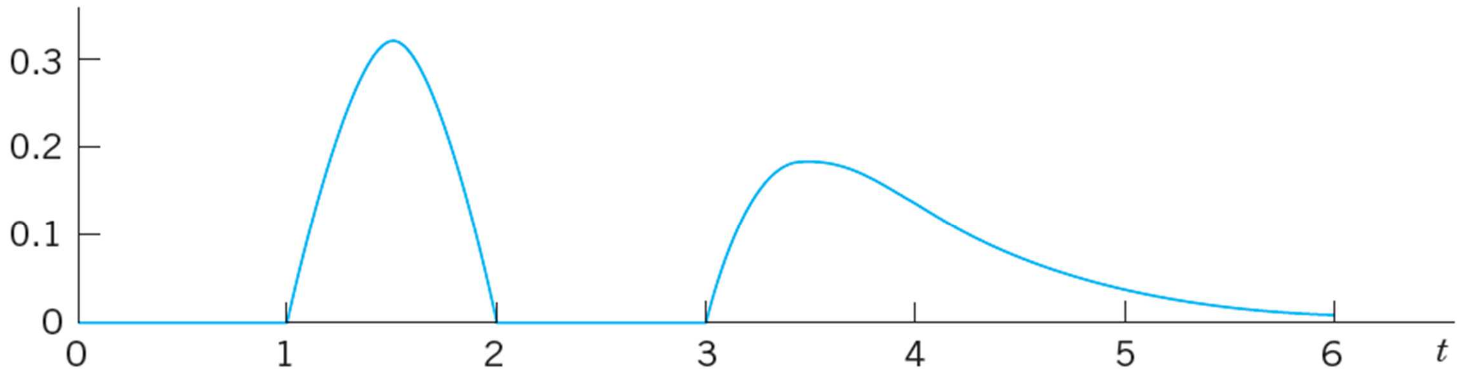

Fig. 123. $f(t)$ in Example 2

6.4 Short Impulses. Dirac's Delta Function. Partial Fractions

Dirac's Delta Function

- An airplane making a "hard" landing, a mechanical system being hit by a hammerblow, a ship being hit by a single high wave, a tennis ball being hit by a racket, and many other similar examples appear in everyday life. They are phenomena of an impulsive nature where actions of forces—mechanical, electrical, etc.—are applied over short intervals of time.
- We can model such phenomena and problems by "Dirac's delta function," and solve them very effectively by the Laplace transform.

Dirac's Delta Function

- To model situations of that type, we consider the function

(1) $$f_k(t-a) = \begin{cases} 1/k & \text{if } a \le t \le a+k \\ 0 & \text{otherwise} \end{cases}$$ (Fig. 132)

(and later its limit as $k \to 0$). This function represents, for instance, a force of magnitude $1/k$ acting from $t = a$ to $t = a + k$, where k is positive and small. In mechanics, the integral of a force acting over a time interval $a \le t \le a + k$ is called the **impulse** of the force; similarly for electromotive forces $E(t)$ acting on circuits. Since the blue rectangle in Fig. 132 has area 1, the impulse of f_k in (1) is

(2) $$I_k = \int_0^\infty f_k(t-a)dt = \int_a^{a+k} \frac{1}{k} dt = 1$$

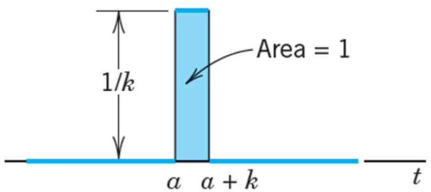

Fig. 132. The function $f_k(t-a)$ in (1)

Dirac's Delta Function

- To find out what will happen if k becomes smaller and smaller, we take the limit of f_k as $k \to 0$ ($k > 0$). This limit is denoted by $\delta(t - a)$, that is,

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a)$$

$\delta(t - a)$ is called the **Dirac delta function** or the **unit impulse function**.

- $\delta(t - a)$ is not a function in the ordinary sense as used in calculus, but a so-called *generalized function*. To see this, we note that the impulse I_k of f_k is 1, so that from (1) and (2) by taking the limit as $k \to 0$ we obtain

(3) $$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases}$$ and $$\int_0^\infty \delta(t-a)\,dt = 1$$

Dirac's Delta Function

- but from calculus we know that a function which is everywhere 0 except at a single point must have the integral equal to 0. Nevertheless, in impulse problems, it is convenient to operate on $\delta(t - a)$ as thought it were an ordinary function. In particular, for a *continuous* function $g(t)$ one uses the property [often called the **sifting property** of $\delta(t - a)$, not to be confused with *shifting*]

(4) $$\int_0^\infty g(t)\delta(t-a)\,dt = g(a)$$

which is plausible by (2).

Dirac's Delta Function

- To obtain the Laplace transform of $\delta(t - a)$ we write

$$f_k(t-a) = \frac{1}{k}[u(t-a) - u(t-(a+k))]$$

and take the transform [see (2)]

$$L\{f_k(t-a)\} = \frac{1}{ks}[e^{-as} - e^{-(a+k)s}] = e^{-as}\frac{1-e^{-ks}}{ks}$$

- We now take the limit as $k \to 0$. By l'Hôpital's rule the quotient on the right has the limit 1 (differentiate the numerator and the denominator separately with respect to k, obtaining se^{-ks} and s, respectively, and use $se^{-ks}/s \to 1$ as $k \to 0$). Hence the right side has the limit e^{-as}. This suggests defining the transform of $\delta(t - a)$ by this limit, that is,

(5) $$L\{\delta(t-a)\} = e^{-as}$$

Example 1

- Determine the response of the damped mass-spring system under a square wave, modeled by

$$y'' + 3y' + 2y = r(t) = u(t-1) - u(t-2) \qquad y(0) = 0, y'(0) = 0$$

- Solution. From (1) and (2) in Sec. 6.2 and (2) and (4) in this section we obtain the subsidiary equation

$$s^2Y + 3sY + 2Y = \tfrac{1}{s}(e^{-s} - e^{-2s}) \qquad Y(s) = \frac{1}{s(s^2+3s+2)}(e^{-s} - e^{-2s})$$

Using the notation *F*(*s*) and partial fractions, we obtain

$$F(s) = \frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

t-shifting: $L\{f(t-a)u(t-a)\} = e^{-as}F(s)$

$$F(s) = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

Example 1

- From Table 6.1 in Sec. 6.1, we see that the inverse is

$$f(t) = L^{-1}(F) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

Therefore, by Theorem 1 in Sec. 6.3 (t-shifting) we obtain the square-wave response shown in Fig. 133,

$$\begin{aligned} y &= L^{-1}(F(s)e^{-s} - F(s)e^{-2s}) \\ &= f(t-1)u(t-1) - f(t-2)u(t-2) \\ &= \begin{cases} 0 & (0 < t < 1) \\ \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)} & (1 < t < 2) \\ -e^{-(t-1)} + e^{-t(t-2)} + \frac{1}{2}e^{-2(t-1)} - \frac{1}{2}e^{-2(t-2)} & (t > 2) \end{cases} \end{aligned}$$

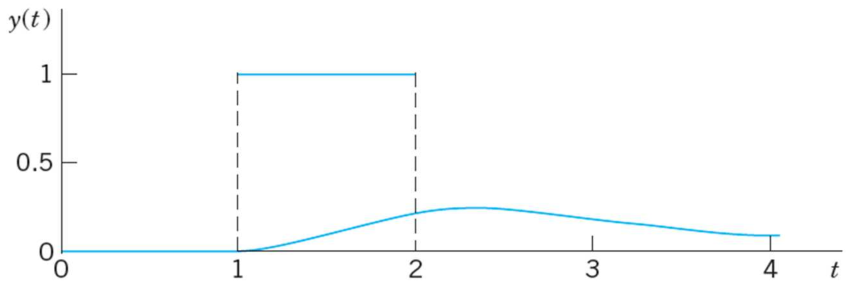

Fig. 133. Square wave and response in Example 1

Example 2

- Find the response of the system in Example 1 with the square wave replaced by a unit impulse at time t = 1.
- Solution. We now have the ODE and the subsidiary equation

$$y'' + 3y' + 2y = \delta(t-1) \qquad (s^2 + 3s + 2)Y = e^{-s}$$

Solving algebraically gives

$$Y(s) = \frac{e^{-s}}{(s+1)(s+2)} = \left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-s}.$$

By Theorem 1 the inverse is

$$y(t) = \mathcal{L}^{-1}(Y) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & \text{if } \quad t > 1. \end{cases}$$

Table 6.1: $e^{at} \longleftrightarrow \dfrac{1}{s-a}$

Example 2

- $y(t)$ is shown in Fig. 134. Can you imagine how Fig. 133 approaches Fig. 134 as the wave becomes shorter and shorter, the area of the rectangle remaining 1?

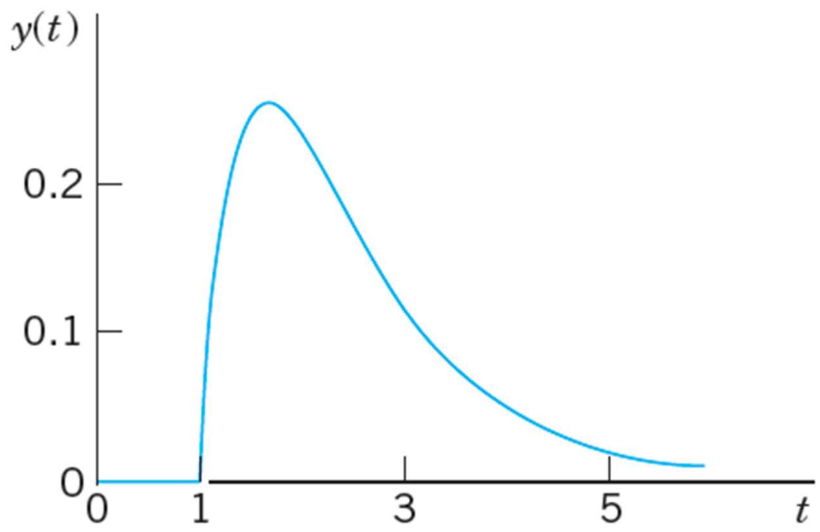

Fig. 134. Response to a hammerblow in Example 2

More on Partial Fractions

- We have seen that the solution Y of a subsidiary equation usually appears as a quotient of polynomials $Y(s) = F(s)/G(s)$, so that a partial fraction representation leads to a sum of expressions whose inverses we can obtain from a table, aided by the first shifting theorem (Sec. 6.1). These representations are sometimes called **Heaviside expansions**.

More on Partial Fractions

- An unrepeated factor s-a in $G(s)$ requires a single partial fraction $A/(s$-$a)$. See Examples 1 and 2. Repeated real factors $(s\text{-}a)^2$, $(s\text{-}a)^3$, etc., require partial fractions

$$\frac{A_2}{(s-a)^2}+\frac{A_1}{s-a} \qquad \frac{A_3}{(s-a)^3}+\frac{A_3}{(s-a)^2}+\frac{A_1}{s-a}$$

The inverses are $(A_2 t + A_1)e^{at}, (\frac{1}{2}A_3 t^2 + A_2 t + A_1)e^{at}$

- Unrepeated complex factors $(s-a)(s-\bar{a}), a=\alpha+i\beta, \bar{a}=\alpha-i\beta$, require a partial fraction $(As+B)/[(s-\alpha)^2+\beta^2]$. For an application, see Example 4 in Sec. 6.3.

Example

- Solve the initial value problem for a damped mass-spring system acted upon by a sinusoidal force for some time interval

$$y'' + 2y' + 2y = r(t), r(t) = 10 \sin 2t \text{ if } 0 < t < \pi \text{ and } 0 \text{ if } t > \pi$$

$$y(0) = 1, y'(0) = -5$$

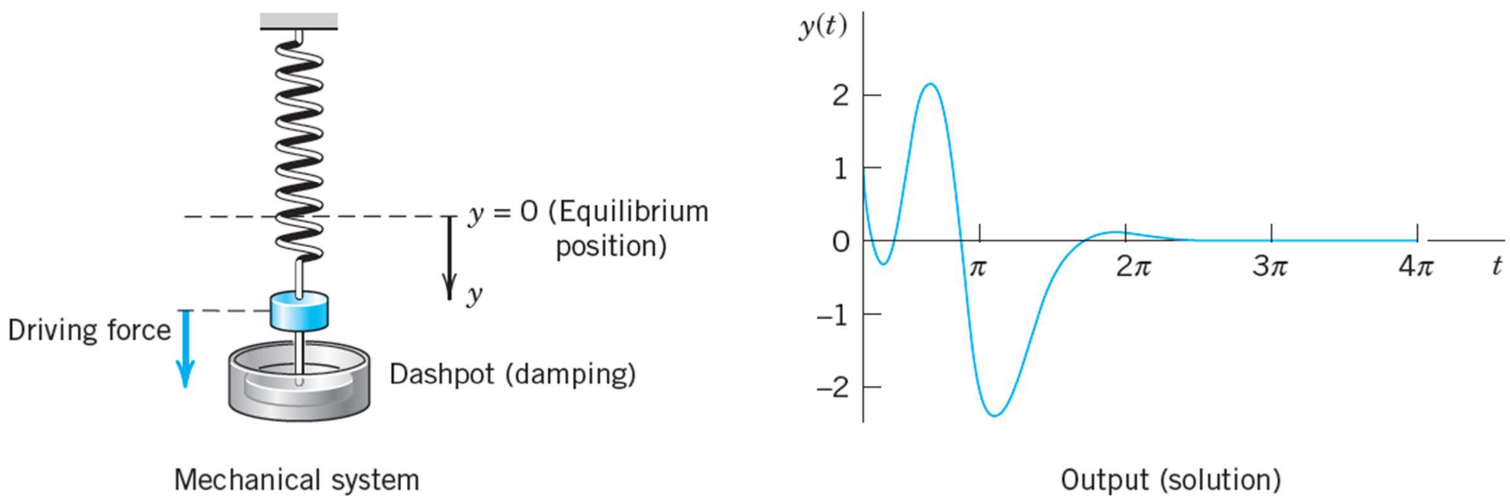

Fig. 136. Example 4

$$y'' + 2y' + 2y = r(t), r(t) = 10\sin 2t \text{ if } 0 < t < \pi \text{ and } 0 \text{ if } t > \pi$$

Example

$$y(0) = 1, y'(0) = -5$$

- From Table 6.1, (1), (2) in Sec. 6.2, and the second shifting theorem in Sec. 6.3, we obtain the subsidiary equation

  $$(s^2Y - s + 5) + 2(sY - 1) + 2Y = 10\frac{2}{s^2+4}(1 - e^{-\pi s})$$

  We collect the Y-terms, $(s^2+2s+2)Y$, take $-s+5-2 = -s+3$ to the right, and solve,

(6) $$Y = \frac{20}{(s^2+4)(s^2+2s+2)} - \frac{20e^{-\pi s}}{(s^2+4)(s^2+2s+2)} + \frac{s-3}{s^2+2s+2}$$

For the last fraction we get from Table 6.1 and the first shifting theorem

(7) $$L^{-1}\left\{\frac{s+1-4}{(s+1)^2+1}\right\} = e^{-t}(\cos t - 4\sin t)$$

Example

- In the first fraction in (6) we have unrepeated complex roots, hence a partial fraction representation

$$\frac{20}{(s^2+4)(s^2+2s+2)} = \frac{As+B}{s^2+4} + \frac{Ms+N}{s^2+2s+2}$$

Multiplication by the common denominator gives

$$20 = (As+B)(s^2+2s+2) + (Ms+N)(s^2+4)$$

We determine A, B, M, N. Equating the coefficients of each power of *s* on both sides gives the four equations

(a) $[s^3] : 0 = A + M$

(b) $[s^2] : 0 = 2A + B + N$

(c) $[s] : 0 = 2A + 2B + 4M$

(d) $[s^0] : 20 = 2B + 4N$

Example

- We can solve this, for instance, obtaining *M*=-*A* from (a), then *A*=*B* from (c), then *N*=-3*A* from (b), and finally *A*=-2 from (d). Hence *A*=-2, *B*=-2, *M*=2, *N*=6, and the first fraction in (6) has the representation

(8) $$\frac{-2s-2}{s^2+4}+\frac{2(s+1)+6-2}{(s+1)^2+1}$$

inverse transform: $-2\cos 2t - \sin 2t + e^{-t}(2\cos t + 4\sin t)$

- The sum of this inverse and (7) is the solution of the problem for $0 < t < \pi$, namely (the sines cancel),

(9) $$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t$$

$$-2\cos 2t - \sin 2t + e^{-t}(2\cos t + 4\sin t)$$

Example

- In the second fraction in (6), taken with the minus sign, we have the factor $e^{-\pi s}$, so that from (8) and the second shifting theorem (Sec. 6.3) we get the inverse transform of this fraction for *t*>0 in the form

$$+2\cos(2t-2\pi)+\sin(2t-2\pi)-e^{-(t-\pi)}[2\cos(t-\pi)+4\sin(t-\pi)]$$
$$=2\cos 2t+\sin 2t+e^{-(t-\pi)}(2\cos t+4\sin t)$$

The sum of this and (9) is the solution for $t > \pi$

(10) $$y(t) = e^{-t}[(3+2e^{\pi})\cos t + 4e^{\pi}\sin t]$$

- Figure 136 shows (9) (for $0 < t < \pi$), a beginning vibration, which goes to zero rapidly because of the damping and the absence of a driving force after $t = \pi$

6.5 Convolution. Integral Equations

Convolution

- Convolution has to do with the multiplication of transforms. The situation is as follows.
- *Addition* of transforms provides no problem; we know that $L(f + g) = L(f) + L(g)$.
- Now **multiplication of transforms** occurs frequently in connection with ODEs, integral equations, and elsewhere. Then we usually know $L(f)$ and $L(g)$ and would like to know the function whose transform is the product $L(f)L(g)$. We might perhaps guess that it is fg, but this is false. *The transform of a product is generally different from the product of the transforms of the factors,*

  $$L(fg) \neq L(f)L(g) \text{ in general.}$$

Convolution

- To see this take $f = e^t$ and $g = 1$. Then $fg = e^t$, $L(fg) = 1/(s - 1)$, but $L(f) = 1/(s - 1)$ and $L(1) = 1/s$ give $L(f)L(g) = 1/(s^2 - s)$.
- According to the next theorem, the correct answer is that $L(f)L(g)$ is the transform of the **convolution** of f and g, denoted by the standard notation $f * g$ and defined by the integral

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t - \tau)d\tau$$

Theorem 1

- **Convolution Theorem**
- *If two functions f and g satisfy the assumption in the existence theorem in Sec*. 6.1, *so that their transforms F and G exist, the product H = FG is the transform of h given by* (1). (Proof after Example 2.)

Example 1

- Let $H(s) = 1/[(s - a)s]$. Find $h(t)$.
- ***Solution:*** $1/(s - a)$ has the inverse $f(t) = e^{at}$ and $1/s$ has the inverse $g(t) = 1$. With $f(\tau) = e^{a\tau}$ and $g(t - \tau) \equiv 1$ we thus obtain from (1) the answer

$$h(t) = e^{at} * 1 = \int_0^t e^{a\tau} \cdot 1 d\tau = \frac{1}{a}(e^{at} - 1)$$

- To check, calculate

$$H(s) = L(h)(s) = \frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right) = \frac{1}{a} \cdot \frac{a}{s^2 - as} = \frac{1}{s-a} \cdot \frac{1}{s} = L(e^{at})L(1)$$

(11) App. 3.1 $\sin x \sin y = \frac{1}{2}[-\cos(x+y) + \cos(x-y)]$

Example 2

- Let $H(s) = 1/(s^2+\omega^2)^2$. Find *h*(*t*).
- Solution. The inverse of $1/(s^2+\omega^2)$ is $(\sin \omega t)/\omega$. Hence from (1) and the first formula in (11) in App. 3.1 we obtain

$$\begin{aligned}
h(t) &= \frac{\sin \omega t}{\omega} * \frac{\sin \omega t}{\omega} = \frac{1}{\omega^2}\int_0^t \sin \omega\tau \sin \omega(t-\tau)d\tau \\
&= \frac{1}{2\omega^2}\int_0^t [-\cos \omega t + \cos(2\omega\tau - \omega t)]d\tau \\
&= \frac{1}{2\omega^2}\left[-\tau \cos \omega t + \frac{\sin \omega\tau}{\omega}\right]_{\tau=0}^{t} \\
&= \frac{1}{2\omega^2}\left[-t \cos \omega t + \frac{\sin \omega t}{\omega}\right]
\end{aligned}$$

in agreement with formula 21 in the table in Sec. 6.9.

Proof of Theorem 1

- CAUTION! Note which ones are the variables of integration! We can denote them as we want, for instance, by τ and *p*, and write

$$F(s) = \int_0^{\infty} e^{-s\tau} f(\tau)d\tau \qquad G(s) = \int_0^{\infty} e^{-sp} g(p)dp$$

We now set $t = p + \tau$, where τ is at first constant. Then $p = t - \tau$, and *t* varies from τ to ∞. Thus

$$G(s) = \int_{\tau}^{\infty} e^{-s(t-\tau)} g(t-\tau)dt = e^{s\tau} \int_{\tau}^{\infty} e^{-st} g(t-\tau)dt$$

τ in *F* and *t* in *G* vary independently.

Proof of Theorem 1

- Hence we can insert the G-integral into the F-integral. Cancellation of $e^{-s\tau}$ and $e^{s\tau}$ then gives

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau) e^{s\tau} \int_\tau^\infty e^{-st} g(t-\tau)\,dt\,d\tau = \int_0^\infty f(\tau) \int_\tau^\infty e^{-st} g(t-\tau)\,dt\,d\tau$$

Here we integrate for fixed τ over t from τ to ∞ and then over τ from 0 to ∞. This is the blue region in Fig. 141.

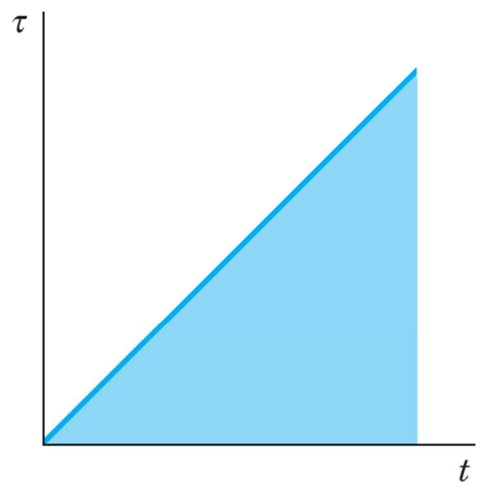

Fig. 141. Region of integration in the $t\tau$-plane in the proof of Theorem 1

Proof of Theorem 1

- Under the assumption on *f* and *g* the order of integration can be reversed. We then integrate first over τ from 0 to *t* and then over *t* from 0 to ∞, that is,

$$F(s)G(s) = \int_0^\infty e^{-st} \int_0^t f(\tau)g(t-\tau)d\tau dt = \int_0^\infty e^{-st} h(t)dt = L(h) = H(s)$$

This completes the proof.

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau) e^{s\tau} \int_\tau^\infty e^{-st} g(t-\tau)dtd\tau = \int_0^\infty f(\tau) \int_\tau^\infty e^{-st} g(t-\tau)dtd\tau$$

Convolution

- Convolution has the properties
- Commutative law: $f*g = g*f$
- Distributive law: $f*(g_1+g_2) = f*g_1 + f*g_2$
- Associative law: $(f*g)*v = f*(g*v)$
- $f*0 = 0*f = 0$

Example 3

- $f * 1 \neq f$ in general. For instance,

$$t * 1 = \int_0^t \tau \cdot 1 d\tau = \frac{1}{2}t^2 \neq t$$

- $(f * f)(t) \geq 0$ may not hold. For instance, Example 2 with $\omega = 1$ gives

$$\sin t * \sin t = -\tfrac{1}{2}t\cos t + \tfrac{1}{2}\sin t$$

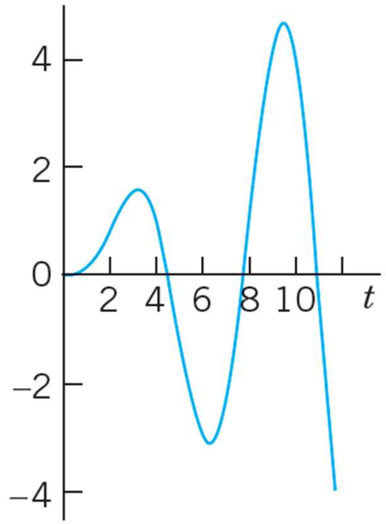

Fig. 142. Example 3

$$h(t) = \frac{\sin\omega t}{\omega} * \frac{\sin\omega t}{\omega} = \frac{1}{\omega^2}\int_0^t \sin\omega\tau \sin\omega(t-\tau)d\tau$$
$$= \frac{1}{2\omega^2}\int_0^t [-\cos\omega t + \cos(2\omega\tau - \omega t)]d\tau$$
$$= \frac{1}{2\omega^2}\left[-\tau\cos\omega t + \frac{\sin\omega\tau}{\omega}\right]_{\tau=0}^{t}$$
$$= \frac{1}{2\omega^2}\left[-t\cos\omega t + \frac{\sin\omega t}{\omega}\right]$$

Example 4

- In an undamped mass-spring system, resonance occurs if the frequency of the driving force equals the natural frequency of the system. Then the model is

$$y'' + \omega_0^2 y = K \sin \omega_0 t$$

where $\omega_0^2 = k/m$, *k* is the spring constant, and *m* is the mass of the body attached to the spring. We assume *y*(0)=0 and *y*'(0)=0, for simplicity. Then the subsidiary equation is

$$s^2 Y + \omega_0^2 Y = \frac{K\omega_0}{s^2 + \omega_0^2} \qquad Y = \frac{K\omega_0}{(s^2 + \omega_0^2)^2}$$

Example 4

- This is a transform as in Example 2 with $\omega = \omega_0$ and multiplied by $K\omega_0$. Hence from Example 2 we can see directly that the solution of our problem is

$$y(t) = \frac{K\omega_0}{2\omega_0^2}\left(-t\cos\omega_0 t + \frac{\sin\omega_0 t}{\omega_0}\right) = \frac{K}{2\omega_0^2}(-\omega_0 t\cos\omega_0 t + \sin\omega_0 t)$$

We see that the first term grows without bound. Clearly, in the case of resonance such a term must occur. (See also a similar kind of solution in Fig. 55 in Sec. 2.8)

$$h(t) = \frac{\sin\omega t}{\omega} * \frac{\sin\omega t}{\omega} = \frac{1}{\omega^2}\int_0^t \sin\omega\tau \sin\omega(t-\tau)d\tau$$

$$= \frac{1}{2\omega^2}\int_0^t [-\cos\omega t + \cos(2\omega\tau - \omega t)]d\tau$$

$$= \frac{1}{2\omega^2}\left[-\tau\cos\omega t + \frac{\sin\omega\tau}{\omega}\right]_{\tau=0}^t$$

$$= \frac{1}{2\omega^2}\left[-t\cos\omega t + \frac{\sin\omega t}{\omega}\right]$$

Application to Nonhomogeneous Linear ODEs

- Nonhomogeneous linear ODEs can now be solved by a general method based on convolution by which the solution is obtained in the form of an integral. To see this, recall from Sec. 6.2 that the subsidiary equation of the ODE

(2) $$y'' + ay' + by = r(t) \qquad (a, b \text{ constant})$$

has the solution [(7) in Sec. 6.2]

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s)$$

with $R(s) = \mathcal{L}(r)$ and $Q(s) = 1/(s^2 + as + b)$ the transfer function.

Application to Nonhomogeneous Linear ODEs

- Inversion of the first term […] provides no difficulty; depending on whether $(1/4)a^2 - b$ is positive, zero, or negative, its inverse will be a linear combination of two exponential functions, or of the form $(c_1 + c_2 t)e^{-at/2}$, or a damped oscillation, respectively. The interesting term is $R(s)Q(s)$ because $r(t)$can have various forms of practical importance, as we shall see. If $y(0) = 0$ and $y'(0) = 0$, then $Y = RQ$, and the convolution theorem gives the solution

(3) $$y(t) = \int_0^t q(t-\tau)r(\tau)\,d\tau$$

Example 5

- Using convolution, determine the response of the damped mass-spring system modeled by

  $$y'' + 3y' + 2y = r(t) \qquad y(0) = y'(0) = 0$$

  $$r(t) = 1 \text{ if } 1 < t < 2 \text{ and } 0 \text{ otherwise}$$

  This system with an input (a driving force) that acts for some time only (Fig. 143) has been solved by partial fraction reduction in Sec. 6.4 (Example 1).

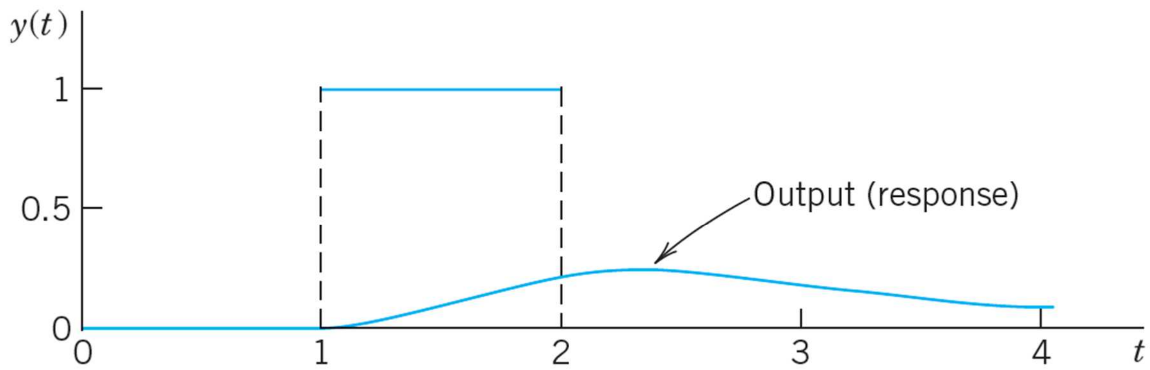

Fig. 143. Square wave and response in Example 5

Example 5

- Solution by Convolution. The transfer function and its inverse are

$$Q(s) = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

hence $q(t) = e^{-t} - e^{-2t}$

- Hence the convolution integral (3) is (except for the limits of integration)

$$y(t) = \int q(t-\tau)\cdot 1 d\tau = \int [e^{-(t-\tau)} - e^{-2(t-\tau)}]d\tau = e^{-(t-\tau)} - \frac{1}{2}e^{-2(t-\tau)}$$

$$y(t) = e^{-(t-\tau)} - \frac{1}{2}e^{-2(t-\tau)}$$

Example 5

- Now comes an important point in handling convolution. $r(\tau) = 1$ if $1 < \tau < 2$ only. Hence if *t*<1, the integral is zero. If 1 < *t* < 2, we have to integrate from $\tau = 1$ (not 0) to *t*. This gives (with the first two terms from the upper limit)

$$y(t) = e^{-0} - \tfrac{1}{2}e^{-0} - \left(e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}\right) = \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)}$$

- If *t*>2, we have to integrate from $\tau = 1$ to 2 (not to *t*). This gives

$$y(t) = e^{-(t-2)} - \tfrac{1}{2}e^{-2(t-2)} - \left(e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}\right)$$

- Figure 143 shows the input (the square wave) and the interesting output, which is zero from 0 to 1, then increases, reaches a maximum (near 2.6) after the input has become zero, and finally decreases to zero in a monotone fashion.

Integral Equations

- Convolution also helps in solving certain **integral equations**, that is, equations in which the unknown function $y(t)$ appears in an integral (and perhaps also outside of it).
- This concerns equations with an integral of the form of a convolution. Hence these are special and it suffices to explain the idea in terms of two examples and add a few problems in the problem set.

Example 6

- Solve the Volterra integral equation of the second kind

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)d\tau = t$$

- Solution. From (1) we see that the given equation can be written as a convolution, $y - y * \sin t = t$. Writing *Y*=*L*(*y*) and applying the convolution theorem, we obtain

$$Y(s) - Y(s)\frac{1}{s^2+1} = Y(s)\frac{s^2}{s^2+1} = \frac{1}{s^2}$$

The solution is

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \qquad y(t) = t + \frac{t^3}{6}$$

Example 7

- Solve the Volterra integral equation

$$y(t) - \int_0^t (1+\tau)y(t-\tau)d\tau = 1 - \sinh t$$

- Solution. By (1) we can write $y - (1+t) * y = 1 - \sinh t$
Writing $Y = L(y)$, we obtain by using the convolution theorem and then taking common denominators

$$Y(s)\left[1 - \left(\frac{1}{s} + \frac{1}{s^2}\right)\right] = \frac{1}{s} - \frac{1}{s^2-1} \qquad Y(s) \cdot \frac{s^2 - s - 1}{s^2} = \frac{s^2 - 1 - s}{s(s^2-1)}$$

$(s^2 - s - 1)/s$ cancels on both sides, so that solving for Y simply gives $Y(s) = \dfrac{s}{s^2-1}$, and the solution is $y(t) = \cosh t$

Table 6.1	$\cosh at \longleftrightarrow \dfrac{s}{s^2 - a^2}$

6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients

Differentiation of Transforms

- It can be shown that, if a function *f*(*t*) satisfies the conditions of the existence theorem in Sec. 6.1, then the derivative *F'*(*s*) = *dF*/*ds* of the transform *F*(*s*) = *L*(*f*) can be obtained by differentiating *F*(*s*) under the integral sign with respect to *s* (proof in Ref. [GenRef4] listed in App. 1). Thus, if

$$F(s) = \int_0^\infty e^{-st} f(t)dt \quad \text{then} \quad F'(s) = -\int_0^\infty e^{-st} t f(t)dt$$

$$F'(s) = -\int_0^{\infty} e^{-st} t f(t) dt$$

Differentiation of Transforms

- Consequently, if $L(f) = F(s)$, then

(1) $L\{tf(t)\} = -F'(s)$, hence $L^{-1}\{F'(s)\} = -tf(t)$

where the second formula is obtained by applying L^{-1} on both sides of the first formula.

- In this way, *differentiation of the transform of a function corresponds to the multiplication of the function by* $-t$.

Example 1

- We shall derive the following three formulas

	$\mathscr{L}(f)$	$f(t)$
(2)	$\dfrac{1}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)$
(3)	$\dfrac{s}{(s^2+\beta^2)^2}$	$\dfrac{t}{2\beta}\sin\beta t$
(4)	$\dfrac{s^2}{(s^2+\beta^2)^2}$	$\dfrac{1}{2\beta}(\sin\beta t + \beta t\cos\beta t)$

$$\text{Table 6.1} \quad \sin \omega t \longleftrightarrow \frac{\omega}{s^2 + \omega^2}$$

Example 1

- Solution. From (1) and formula 8 (with $\omega = \beta$) in Table 6.1 of Sec. 6.1 we obtain by differentiation (CAUTION! Chain rule!)

$$L(t \sin \beta t) = \frac{2\beta s}{(s^2 + \beta^2)^2}$$

Dividing by 2β and using the linearity of L, we obtain (3).

$$\sin \beta t \longleftrightarrow \frac{\beta}{s^2 + \beta^2}$$

$$t \sin \beta t \longleftrightarrow -\left(\frac{\beta}{s^2 + \beta^2}\right)' = -\left(\frac{-2\beta s}{(s^2 + \beta^2)^2}\right)$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \qquad u = \beta \qquad v = s^2 + \beta^2$$

Table 6.1 $\cos \omega t \longleftrightarrow \dfrac{s}{s^2+\omega^2}$

Example 1

- Formulas (2) and (4) are obtained as follows. From (1) and formula 7 (with $\omega = \beta$) in Table 6.1 we find

$$L(t\cos\beta t) = -\frac{(s^2+\beta^2)-2s^2}{(s^2+\beta^2)^2} = \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

From this and formula 8 (with $\omega = \beta$) in Table 6.1 we have

$$L\left(t\cos\beta t \pm \frac{1}{\beta}\sin\beta t\right) = \frac{s^2-\beta^2}{(s^2+\beta^2)^2} \pm \frac{1}{s^2+\beta^2}$$

On the right we now take the common denominator. Then we see that for the plus sign the numerator becomes $s^2-\beta^2+s^2+\beta^2=2s^2$, so that (4) follows by division by 2. Similarly, for the minus sign the numerator takes the form $s^2-\beta^2-s^2-\beta^2=-2\beta^2$, and we obtain (2).

Integration of Transforms

- Similarly, if $f(t)$ satisfies the conditions of the existence theorem in Sec. 6.1 and the limit of $f(t)/t$ as t approaches 0 from the right, exists, then for $s > k$,

(6) $$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})d\tilde{s} \qquad L^{-1}\left\{\int_s^\infty F(\tilde{s})d\tilde{s}\right\} = \frac{f(t)}{t}$$

In this way, integration of the transform of a function $f(t)$ corresponds to the division of $f(t)$ by t.

- We indicate how (6) is obtained. From the definition it follows that

$$\int_s^\infty F(\tilde{s})d\tilde{s} = \int_s^\infty \left[\int_0^\infty e^{-\tilde{s}t} f(t)dt\right] d\tilde{s}$$

Integration of Transforms

- And it can be shown that under the above assumptions we may reverse the order of integration, that is,

$$\int_s^\infty F(\tilde{s})d\tilde{s} = \int_0^\infty \left[\int_s^\infty e^{-\tilde{s}t} f(t) d\tilde{s}\right] dt = \int_0^\infty f(t) \left[\int_s^\infty e^{-\tilde{s}t} d\tilde{s}\right] dt$$

Integration of $e^{-\tilde{s}t}$ with respect to $\tilde{s}$ gives $e^{-\tilde{s}t}/(-t)$. Here the integral over $\tilde{s}$ on the right equals e^{-st}/t . Therefore,

$$\int_s^\infty F(\tilde{s})d\tilde{s} = \int_0^\infty e^{-st} \frac{f(t)}{t} dt = L\left\{\frac{f(t)}{t}\right\}$$

$$\left.\frac{e^{-\tilde{s}t}}{-t}\right|_s^\infty$$

$$(\ln x)' = \frac{1}{x}$$

Example 2

- Find the inverse transform of $\ln\left(1+\frac{\omega^2}{s^2}\right) = \ln\frac{s^2+\omega^2}{s^2}$
- Solution. Denote the given transform by *F*(*s*). Its derivative is

$$F'(s) = \frac{d}{ds}(\ln(s^2+\omega^2) - \ln s^2) = \frac{2s}{s^2+\omega^2} - \frac{2s}{s^2}$$

Taking the inverse transform and using (1), we obtain

$$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right\} = 2\cos\omega t - 2 = -tf(t)$$

Hence the inverse *f*(*t*) of *F*(*s*) is $f(t) = 2(1-\cos\omega t)/t$
This agrees with formula 42 in Sec. 6.9.

$$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{2s}{s^2+\omega^2} - \frac{2}{s}\right\} = 2\cos\omega t - 2 = -tf(t)$$

Example 2

$$(6) \quad L^{-1}\left\{\int_s^\infty F(\tilde{s})d\tilde{s}\right\} = \frac{f(t)}{t}$$

- Alternatively, if we let

$$G(s) = \frac{2s}{s^2+\omega^2} - \frac{2}{s} \qquad g(t) = L^{-1}(G) = 2(\cos\omega t - 1)$$

From this and (6) we get, in agreement with the answer just obtained

$$L^{-1}\left\{\ln\frac{s^2+\omega^2}{s^2}\right\} = L^{-1}\left\{\int_s^\infty G(s)ds\right\} = -\frac{g(t)}{t} = \frac{2}{t}(1-\cos\omega t)$$

the minus occurring since s is the lower limit of integration.

- In a similar way we obtain formula 43 in Sec. 6.9

$$L^{-1}\left\{\ln\left(1-\frac{a^2}{s^2}\right)\right\} = \frac{2}{t}(1-\cosh at)$$

Special Linear ODEs with Variable Coefficients

- Formula (1) can be used to solve certain ODEs with variable coefficients. The idea is this. Let $L(y) = Y$. Then $L(y') = sY - y(0)$ (see Sec. 6.2). Hence by (1),

(7) $$L(ty') = -\frac{d}{ds}[sY - y(0)] = -Y - s\frac{dY}{ds}$$

Similarly, $L(y'') = s^2Y - sy(0) - y'(0)$ and by (1)

(8) $$L(ty'') = -\frac{d}{ds}[s^2Y - sy(0) - y'(0)] = -2sY - s^2\frac{dY}{ds} + y(0)$$

(1) $$L\{tf(t)\} = -F'(s)$$

$$(uv)' = u'v + uv'$$

Special Linear ODEs with Variable Coefficients

- Hence if an ODE has coefficients such as $at+b$, the subsidiary equation is a first-order ODE for Y, which is sometimes simpler than the given second-order ODE.
- But if the latter has coefficients $at^2 + bt + c$, then two applications of (1) would give a second-order ODE for Y, and this shows that the present method works well only for rather special ODEs with variable coefficients.

Example 3

- Laguerre's ODE is

(9) $$ty'' + (1-t)y' + ny = 0$$

We determine a solution of (9) with *n*=0,1, …. From (7)-(9) we get the subsidiary equation

$$\left[-2sY - s^2\frac{dY}{ds} + y(0)\right] + sY - y(0) - \left(-Y - s\frac{dY}{ds}\right) + nY = 0$$

Simplification gives

$$(s - s^2)\frac{dY}{ds} + (n+1-s)Y = 0$$

(7) $$L(ty') = -Y - s\frac{dY}{ds}$$

(8) $$L(ty'') = -2sY - s^2\frac{dY}{ds} + y(0)$$

Example 3

- Separating variables, using partial fractions, integrating (with the constant of integration taken to be zero), and taking exponentials, we get

(10*) $$\frac{dY}{Y} = -\frac{n+1-s}{s-s^2}ds = \left(\frac{n}{s-1} - \frac{n+1}{s}\right)ds \qquad Y = \frac{(s-1)^n}{s^{n+1}}$$

We write $l_n = L^{-1}(Y)$ and prove **Rodrigues's formula**

(10) $$l_0 = 1, \quad l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}) \qquad n = 1, 2, \ldots$$

These are polynomials because the exponential terms cancel if we perform the indicated differentiations.

Example 3

- They are called **Laguerre polynomials** and are usually denoted by L_n. We prove (10). By Table 6.1 and the first shifting theorem (*s*-shifting),

$$L(t^n e^{-t}) = \frac{n!}{(s+1)^{n+1}}$$

hence by (3) in Sec. 6.2 $L\left\{\frac{d^n}{dt^n}(t^n e^{-t})\right\} = \frac{n! s^n}{(s+1)^{n+1}}$

because the derivatives up to the order *n*-1 are zero at 0. Now make another shift and divide by *n*! to get

$$L(l_n) = \frac{(s-1)^n}{s^{n+1}} = Y$$

$$L(f^{(n)}) = s^n L(F) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

6.7 System of ODEs

System of ODEs

- The Laplace transform method may also be used for solving systems of ODEs, as we shall explain in terms of typical applications. We consider a first-order linear system with constant coefficients (as discussed in Sec. 4.1)

$$(1) \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + g_1(t) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + g_2(t) \end{aligned}$$

- Writing $Y_1 = L(y_1)$, $Y_2 = L(y_2)$, $G_1 = L(g_1)$, $G_2 = L(g_2)$, we obtain from (1) in Sec. 6.2 the subsidiary system

$$sY_1 - y_1(0) = a_{11}Y_1 + a_{12}Y_2 + G_1(s)$$

$$sY_2 - y_2(0) = a_{21}Y_1 + a_{22}Y_2 + G_2(s)$$

System of ODEs

- By collecting the Y_1- and Y_2-terms we have

$$(a_{11} - s)Y_1 + a_{12}Y_2 = -y_1(0) - G_1(s)$$

(2)
$$a_{21}Y_1 + (a_{22} - s)Y_2 = -y_2(0) - G_2(s)$$

- By solving this system algebraically for $Y_1(s)$, $Y_2(s)$ and taking the inverse transform we obtain the solution $y_1 = L^{-1}(Y_1)$, $y_2 = L^{-1}(Y_2)$ of the given system (1). Note that (1) and (2) may be written in vector form (and similarly for the systems in the examples); thus, setting $\mathbf{y} = [y_1 \ \ y_2]^T$, $\mathbf{A} = [a_{jk}]$, $\mathbf{g} = [g_1 \ \ g_2]^T$, $\mathbf{Y} = [Y_1 \ \ Y_2]^T$, $\mathbf{G} = [G_1 \ \ G_2]^T$ we have

$$\mathbf{y}' = \mathbf{Ay} + \mathbf{g} \quad \text{and} \quad (\mathbf{A} - s\mathbf{I})\mathbf{Y} = -\mathbf{y}(0) - \mathbf{G}.$$

Example 1

- Tank T_1 in Fig. 144 initially contains 100 gal of pure water. Tank T_2 initially contains 100 gal of water in which 150 lb of salt are dissolved. The inflow into T_1 is 2 gal/min from T_2 and 6 gal/min containing 6 lb of salt from the outside. The inflow into T_2 is 8 gal/min from T_1. The outflow from T_2 is 2 + 6 = 8 gal/min, as shown in the figure. The mixtures are kept uniform by stirring. Find and plot the salt contents $y_1(t)$ and $y_2(t)$ in T_1 and T_2, respectively.

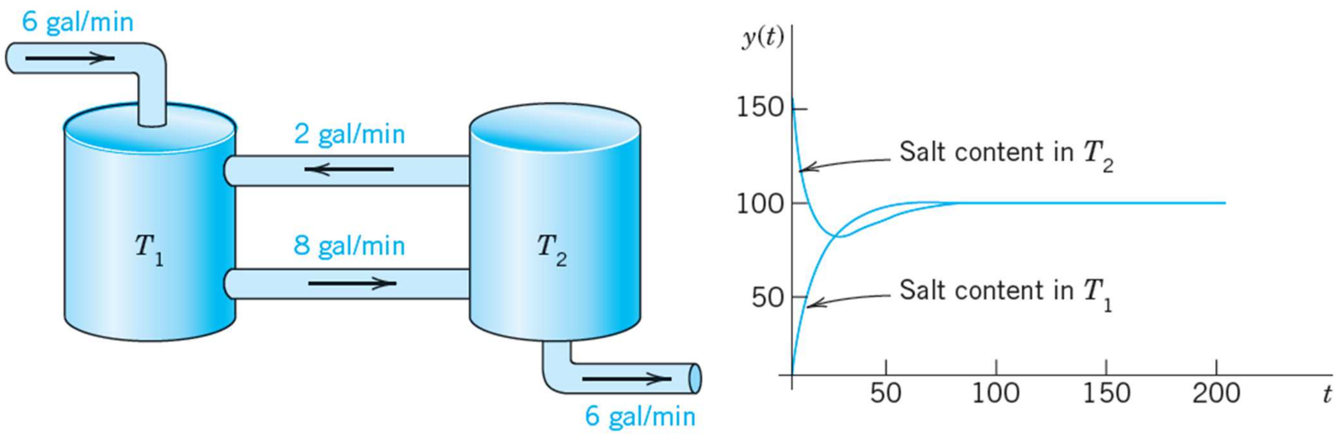

Fig. 144. Mixing problem in Example 1

Example 1

- Solution. The model is obtained in the form of two equations

Time rate of change = Inflow/min − Outflow/min

for the two tanks (see Sec. 4.1). Thus,

$$y_1' = -\frac{8}{100}y_1 + \frac{2}{100}y_2 + 6 \qquad y_2' = \frac{8}{100}y_1 - \frac{8}{100}y_2$$

- The initial conditions are $y_1(0) = 0$, $y_2(0) = 150$. From this we see that the subsidiary system (2) is

$$(-0.08 - s)Y_1 + 0.02Y_2 = -\frac{6}{s}$$

$$0.08Y_1 + (-0.08 - s)Y_2 = -150$$

Example 1

- We solve this algebraically for Y_1 and Y_2 by elimination (or by Cramer's rule in Sec. 7.7), and we write the solutions in terms of partial fractions,

$$Y_1 = \frac{9s + 0.48}{s(s + 0.12)(s + 0.04)} = \frac{100}{s} - \frac{62.5}{s + 0.12} - \frac{37.5}{s + 0.04}$$

$$Y_2 = \frac{150s^2 + 12s + 0.48}{s(s + 0.12)(s + 0.04)} = \frac{100}{s} + \frac{125}{s + 0.12} - \frac{75}{s + 0.04}$$

Example 1

- By taking the inverse transform we arrive at the solution

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$
$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}.$$

- Figure 144 shows the interesting plot of these functions. Can you give physical explanations for their main features? Why do they have the limit 100? Why is y_2 not monotone, whereas y_1 is? Why is y_1 from some time on suddenly larger than y_2? Etc.

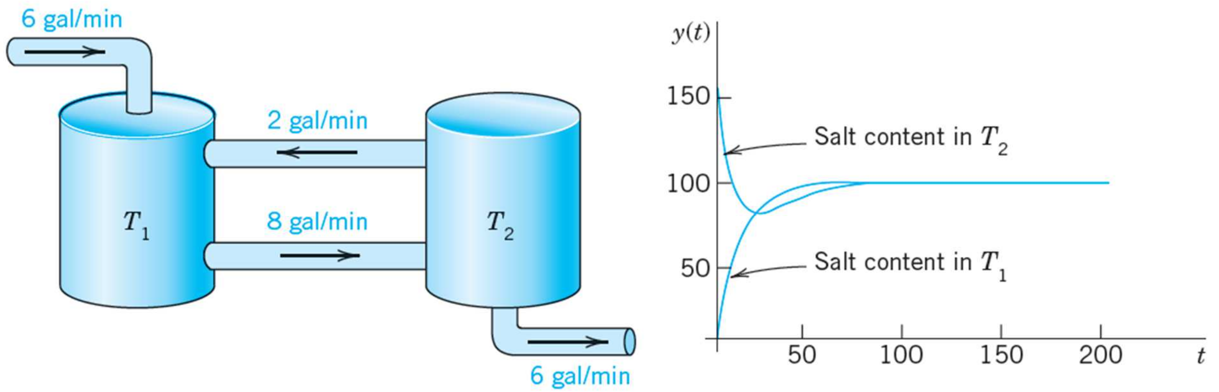

Fig. 144. Mixing problem in Example 1

Example 3

- The mechanical system in Fig. 146 consists of two bodies of mass 1 on three springs of the same spring constant *k* and of negligibly small masses of the springs. Also damping is assumed to be practically zero. Then the model of the physical system is the system of ODEs

$$\text{(3)} \qquad \begin{aligned} y_1'' &= -ky_1 + k(y_2 - y_1) \\ y_2'' &= -k(y_2 - y_1) - ky_2 \end{aligned}$$

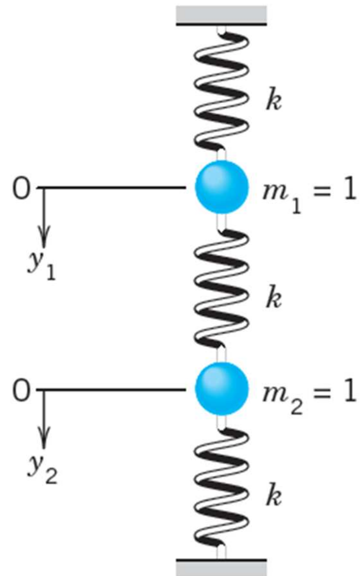

Fig. 146. Example 3

Example 3

- Here y_1 and y_2 are the displacements of the bodies from their positions of static equilibrium. These ODEs follow from Newton's second law, *Mass* X *Acceleration* = *Force*, as in Sec. 2.4 for a single body. We again regard downward forces as positive and upward as negative. On the upper body, $-ky_1$ is the force of the upper spring and $k(y_2-y_1)$ that of the middle spring, y_2-y_1 being the net change in spring length – think this over before going on. On the lower body, $-k(y_2-y_1)$ is the force of the middle spring and $-k_2$ that of the lower spring.

Example 3

$$y_1'' = -ky_1 + k(y_2 - y_1)$$
$$y_2'' = -k(y_2 - y_1) - ky_2$$

- We shall determine the solution corresponding to the initial conditions $y_1(0) = 1, y_2(0) = 1, y_1'(0) = \sqrt{3k}, y_2'(0) = -\sqrt{3k}$ Let $Y_1 = L(y_1), Y_2 = L(y_2)$. Then from (2) in Sec. 6.2 and the initial conditions we obtain the subsidiary system

$$s^2 Y_1 - s - \sqrt{3k} = -kY_1 + k(Y_2 - Y_1)$$
$$s^2 Y_2 - s + \sqrt{3k} = -k(Y_2 - Y_1) - kY_2$$

This system of linear algebraic equations in the unknowns Y_1 and Y_2 may be written

$$(s^2 + 2k)Y_1 - kY_2 = s + \sqrt{3k}$$
$$-ky_1 + (s^2 + 2k)Y_2 = s - \sqrt{3k}$$

Example 3

- Elimination (or Cramer's rule in Sec. 7.7) yields the solutions, which we can expand in terms of partial fractions,

$$Y_1 = \frac{(s+\sqrt{3k})(s^2+2k)+k(s-\sqrt{3k})}{(s^2+2k)^2-k^2} = \frac{s}{s^2+k} + \frac{\sqrt{3k}}{s^2+3k}$$

$$Y_2 = \frac{(s^2+2k)(s-\sqrt{3k})+k(s+\sqrt{3k})}{(s^2+2k)^2-k^2} = \frac{s}{s^2+k} - \frac{\sqrt{3k}}{s^2+3k}$$

Here the solution of our initial value problem is (Fig. 147)

$$y_1(t) = L^{-1}(Y_1) = \cos\sqrt{k}t + \sin\sqrt{3k}t$$
$$y_2(t) = L^{-1}(Y_2) = \cos\sqrt{k}t - \sin\sqrt{3k}t$$

Example 3

- We see that the motion of each mass is harmonic (the system is undamped!), being the superposition of a "slow" oscillation and a "rapid" oscillation.

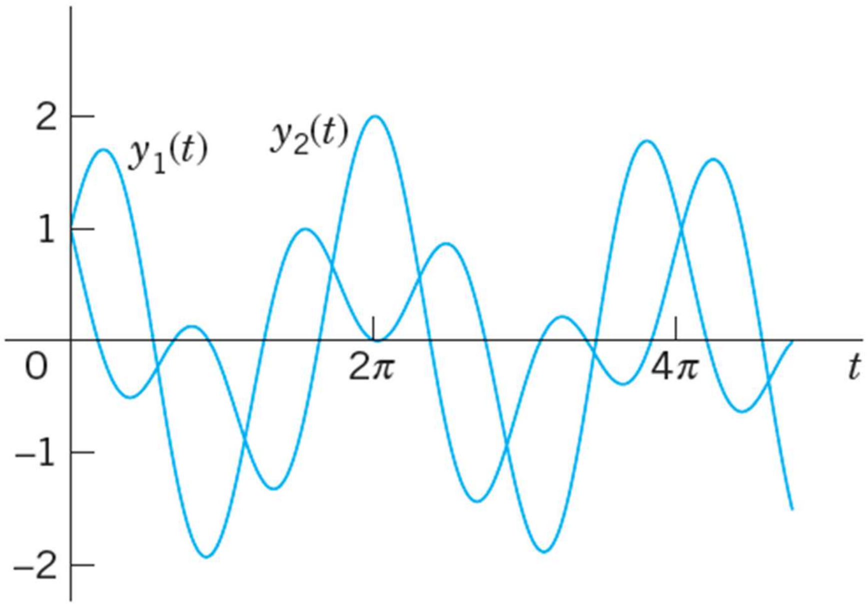

Fig. 147. Solutions in Example 3

Summary of Chapter 6

Summary of Chapter 6

- The main purpose of Laplace transforms is the solution of differential equations and systems of such equations, as well as corresponding initial value problems. The **Laplace transform** $F(s) = L(f)$ of a function $f(t)$ is defined by

(1) $$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt.$$ (Sec. 6.1).

- This definition is motivated by the property that the differentiation of f with respect to t corresponds to the multiplication of the transform F by s; more precisely,

(2) $$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$ $$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$ (Sec. 6.2)

Summary of Chapter 6

- Hence by taking the transform of a given differential equation

(3) $y'' + ay' + by = r(t)$ (a, b constant)

and writing L $(y) = Y(s)$, we obtain the **subsidiary equation**

(4) $(s^2 + as + b)Y = \text{L}\,(r) + sf(0) + f\,'(0) + af\,(0)$.

- Here, in obtaining the transform $L(r)$ we can get help from the small table in Sec. 6.1 or the larger table in Sec. 6.9. This is the first step. In the second step we solve the subsidiary equation *algebraically* for $Y(s)$.

Summary of Chapter 6

- In the third step we determine the **inverse transform** $y(t) = L^{-1}(Y)$, that is, the solution of the problem. This is generally the hardest step, and in it we may again use one of those two tables. $Y(s)$ will often be a rational function, so that we can obtain the inverse $L^{-1}(Y)$ by partial fraction reduction (Sec. 6.4) if we see no simpler way.
- The Laplace method avoids the determination of a general solution of the homogeneous ODE, and we also need not determine values of arbitrary constants in a general solution from initial conditions; instead, we can insert the latter directly into (4).

Summary of Chapter 6

- Two further facts account for the practical importance of the Laplace transform. First, it has some basic properties and resulting techniques that simplify the determination of transforms and inverses. The most important of these properties are listed in Sec. 6.8, together with references to the corresponding sections. More on the use of unit step functions and Dirac's delta can be found in Secs. 6.3 and 6.4, and more on convolution in Sec. 6.5.

Summary of Chapter 6

- Second, due to these properties, the present method is particularly suitable for handling right sides $r(t)$ given by different expressions over different intervals of time, for instance, when $r(t)$ is a square wave or an impulse or of a form such as $r(t) = \cos t$ if $0 \leq t \leq 4\pi$ and 0 elsewhere.
- The application of the Laplace transform to systems of ODEs is shown in Sec. 6.7. (The application to PDEs follows in Sec. 12.12.)