Chapter 11 Fourier Analysis

Advanced Engineering Mathematics

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11.1 Fourier Series
Fourier Series

- Fourier series are infinite series that represent periodic functions in terms of cosines and sines. As such, Fourier series are of greatest importance to the engineer and applied mathematician.

- A function \( f(x) \) is called a **periodic function** if \( f(x) \) is defined for all real \( x \), except possibly at some points, and if there is some positive number \( p \), called a **period** of \( f(x) \), such that

\[
(1) \quad f(x + p) = f(x)
\]
Fourier Series

- The graph of a periodic function has the characteristic that it can be obtained by periodic repetition of its graph in any interval of length \( p \). (Fig. 258)
- The smallest positive period is often called the \textit{fundamental period}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fourier_series_graph.png}
\caption{Periodic function of period \( p \)}
\end{figure}
Fourier Series

- If $f(x)$ has period $p$, it also has the period $2p$ because (1) implies $f(x + 2p) = f([x + p] + p) = f(x + p) = f(x)$; thus for any integer $n=1, 2, 3, \ldots$

\[
f(x + np) = f(x)
\]

Furthermore if $f(x)$ and $g(x)$ have period $p$, then $af(x) + bg(x)$ with any constants $a$ and $b$ also has the period $p$. 
Fourier Series

• Our problem in the first few sections of this chapter will be the representation of various functions $f(x)$ of period $2\pi$ in terms of the simple functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, ..., \cos nx, \sin nx, ...$$

• All these functions have the period $2\pi$. They form the so-called trigonometric system. Figure 259 shows the first few of them.

![Cosine and sine functions](image)

**Fig. 259.** Cosine and sine functions having the period $2\pi$ (the first few members of the trigonometric system (3), except for the constant 1)
Fourier Series

• The series to be obtained will be a trigonometric series, that is, a series of the form

\[
a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots
\]

\[
= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

\(a_0, a_1, b_1, a_2, b_2, \ldots\) are constants, called the coefficients of the series. We see that each term has the period \(2\pi\). Hence if the coefficients are such that the series converges, its sum will be a function of period \(2\pi\).
Fourier Series

Now suppose that \( f(x) \) is a given function of period \( 2\pi \) and is such that it can be represented by a series (4), that is, (4) converges and, moreover, has the sum \( f(x) \). Then, using the equality sign, we write

\[
f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

and call (5) the Fourier series of \( f(x) \). We shall prove that in this case the coefficients of (5) are the so-called Fourier coefficients of \( f(x) \), given by the Euler formulas

\[
(6.0) \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx
\]

\[
(6.a) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx
\]

\[
(6.b) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx
\]
Example 1

- Find the Fourier coefficients of the period function $f(x)$ in Fig. 260. The formula is

$$f(x) = \begin{cases} -k & \text{if} \ -\pi < x < 0 \\ k & \text{if} \ 0 < x < \pi \end{cases} \quad f(x + 2\pi) = f(x)$$

Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$

Fig. 260. Given function $f(x)$ (Periodic rectangular wave)
Example 1

- From (6.0) we obtain \( a_0 = 0 \). This can also be seen without integration, since the area under the curve of \( f(x) \) between \(-\pi\) and \( \pi \) is zero. From (6.a) we obtain the coefficient \( a_1, a_2, \ldots \) of the cosine terms. Since \( f(x) \) is given by two expressions, the integrals from \(-\pi\) to \( \pi \) split into two integrals:

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} (-k) \cos nx \, dx + \int_{0}^{\pi} k \cos nx \, dx \right]
\]

\[
= \frac{1}{\pi} \left[ k \sin nx \bigg|_{0}^{\pi} n - k \sin nx \bigg|_{0}^{-\pi} \right] = 0
\]

because \( \sin nx = 0 \) at \(-\pi, 0, \pi\) for all \( n = 1, 2, \ldots \).
Example 1

- We see that all these cosine coefficients are zero. That is, the Fourier series of (7) has no cosine terms, just sine terms, it is a Fourier sine series with coefficients $b_1, b_2, \ldots$ obtained from (6b);

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} (-k) \sin nx \, dx + \int_{0}^{\pi} k \sin nx \, dx \right]
\]

\[
= \frac{1}{\pi} \left[ k \frac{\cos nx}{n} \bigg|_{-\pi}^{0} - k \frac{\cos nx}{n} \bigg|_{0}^{\pi} \right]
\]

Since $\cos(-\alpha) = \cos \alpha$, $\cos 0 = 1$, this yields

\[
b_n = \frac{k}{n\pi} [\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{n\pi} (1 - \cos n\pi)
\]
Example 1

- Now, \( \cos \pi = -1, \cos 2\pi = 1, \cos 3\pi = -1, \text{ etc.}; \) in general,

\[
\cos n\pi = \begin{cases} 
-1 & \text{for odd } n \\
1 & \text{for even } n
\end{cases}
\]

\[
1 - \cos n\pi = \begin{cases} 
2 & \text{for odd } n \\
0 & \text{for even } n
\end{cases}
\]

Hence the Fourier coefficients \( b_n \) of our function are

\[
b_1 = \frac{4k}{\pi}, b_2 = 0, b_3 = \frac{4k}{3\pi}, b_4 = 0, b_5 = \frac{4k}{5\pi}, \ldots
\]

\[
b_n = \frac{k}{n\pi} \left[ \cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0 \right] = \frac{2k}{n\pi} (1 - \cos n\pi)
\]
Example 1

- Since the $a_n$ are zero, the Fourier series of $f(x)$ is

$$ f(x) \approx \frac{4k}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right) \quad (8) $$

The partial sums are

$$ S_1 = \frac{4k}{\pi} \sin x \quad S_2 = \frac{4k}{\pi} \left( \sin x + \frac{1}{3} \sin 3x \right) $$

Their graphs in Fig. 261 seem to indicate that the series is convergent and has the sum $f(x)$, the given function. We notice that at $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the limits $-k$ and $k$ of our function, at these points. This is typical.
Example 1

Furthermore, assuming that \( f(x) \) is the sum of the series and setting \( x = \pi/2 \), we have

\[
f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right)
\]

Thus \( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4} \)

This is a famous result obtained by Leibniz in 1673 from geometric considerations. It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points.
Example 1

Fig. 261. First three partial sums of the corresponding Fourier series
Theorem 1 Orthogonality of the Trigonometric System

- The trigonometric system (3) is orthogonal on the interval $-\pi \leq x \leq \pi$ (hence also on $0 \leq x \leq 2\pi$ or any other interval of length $2\pi$ because of periodicity); that is, the integral of the product of any two functions in (3) over that interval is 0, so that for any integers $n$ and $m$,

(a) $\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \quad (n \neq m)$

(b) $\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \quad (n \neq m)$

(c) $\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \quad (n \neq m \text{ or } n = m)$

(3) $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots$
Proof of Theorem 1

- Simply by transforming the integrands trigonometrically from products into sums

\[ \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n + m) \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos(n - m) \, dx \]

\[ \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n - m) \, dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos(n + m) \, dx \]

Since \( m \neq n \) (integer!), the integrals on the right are all 0. Similarly, for all integers

\[ \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin(n + m) \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin(n - m) \, dx = 0 + 0 \]

\[
\begin{align*}
\cos x \cos y &= \frac{1}{2} [\cos(x + y) + \cos(x - y)] \\
\sin x \sin y &= \frac{1}{2} [-\cos(x + y) + \cos(x - y)] \\
\sin x \cos y &= \frac{1}{2} [\sin(x + y) + \sin(x - y)]
\end{align*}
\]
Application of Theorem 1 to the Fourier Series (5)

- We prove (6.0). Integrating on both sides of (5) from $-\pi$ to $\pi$, we get

\[ \int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \left[ a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] dx \]

We now assume that termwise integration is allowed. Then we obtain

\[ \int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right) \]

The first term on the right equals $2\pi a_0$. Integration shows that all the other integrals are 0. Hence division by $2\pi$ gives (6.0).

\[ (6.0) \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \]
Application of Theorem 1 to the Fourier Series (5)

(6.a) \( a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \)

- We prove (6.a). Multiplying (5) on both sides by \( \cos mx \) with any fixed positive integer \( m \) and integrating from \( -\pi \) to \( \pi \), we have

\[
\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi} \left[ a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \cos mx \, dx
\]

We now integrate term by term. Then on the right we obtain an integral of \( a_0 \cos mx \), which is 0; an integral of \( a_n \cos nx \cos mx \), which is \( a_m \pi \) for \( n = m \) and 0 for \( n \neq m \) by (9a); and an integral of \( b_n \sin nx \cos mx \), which is 0 for all \( n \) and \( m \) by (9c). Hence the right side of (10) equals \( a_m \pi \). Division by \( \pi \) gives (6a) (with \( m \) instead of \( n \)).

\[
\int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{4} \sin 2x + c
\]
Application of Theorem 1 to the Fourier Series (5)

(6.b) \[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \]

- We finally prove (6.b). Multiplying (5) on both sides by \( \sin mx \) with any fixed positive integer \( m \) and integrating from \( -\pi \) to \( \pi \), we get

\[ \int_{-\pi}^{\pi} f(x) \sin mx \, dx = \int_{-\pi}^{\pi} \left[ a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \sin mx \, dx \]

Integrating term by term, we obtain on the right an integral of \( a_0 \sin mx \), which is 0; an integral of \( a_n \cos nx \sin mx \), which is 0 by (9c); and an integral of \( b_n \sin nx \sin mx \), which is \( b_m \pi \) if \( n = m \) and 0 if \( n \neq m \), by (9b). This implies (6b) (with \( n \) denoted by \( m \)). This completes the proof of the Euler formulas (6) for the Fourier coefficients.

\[ \int \sin^2 x \, dx = \frac{1}{2} x - \frac{1}{4} \sin 2x + c \]
Convergence and Sum of a Fourier Series

- The class of functions that can be represented by Fourier series is surprisingly large and general.
- Theorem 2: Let $f(x)$ be periodic with period $2\pi$ and piecewise continuous in the interval $-\pi \leq x \leq \pi$. Furthermore, let $f(x)$ have a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series (5) of $f(x)$ [with coefficients (6)] converges. Its sum is $f(x)$, except at points $x_0$ where $f(x)$ is discontinuous. There the sum of the series is the average of the left- and right-hand limits of $f(x)$ at $x_0$. 
The left-hand limit of \( f(x) \) at \( x_0 \) is defined as the limit of \( f(x) \) as \( x \) approaches \( x_0 \) from the left and is commonly denoted by \( f(x_0^-) \). Thus

\[
f(x_0^-) = \lim_{h \to 0} f(x_0 - h) \quad \text{as} \quad h \to 0 \quad \text{through positive values.}
\]

The right-hand limit is denoted by \( f(x_0^+) \) and

\[
f(x_0^+) = \lim_{h \to 0} f(x_0 + h) \quad \text{as} \quad h \to 0 \quad \text{through positive values.}
\]

The left- and right-hand derivatives of \( f(x) \) at \( x_0 \) are defined as the limits of

\[
\frac{f(x_0+h) - f(x_0)}{-h} \quad \text{and} \quad \frac{f(x_0+h) - f(x_0)}{-h}
\]

respectively, as \( h \to 0 \) through positive values. Of course if \( f(x) \) is continuous at \( x_0 \), the last term in both numerators is simply \( f(x_0) \).
Example 2

- The rectangular wave in Example 1 has a jump at $x = 0$. Its left-hand limit there is $-k$ and its right-hand limit is $k$ (Fig. 261). Hence the average of these limits is 0. The Fourier series (8) of the wave does indeed converge to this value when $x=0$ because then all its terms are 0. Similarly for the other jumps. This is in agreement with Theorem 2.
Summary

- A Fourier series of a given function $f(x)$ of period $2\pi$ is a series of the form (5) with coefficients given by the Euler formulas (6). Theorem 2 gives conditions that are sufficient for this series to converge and at each $x$ to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand and right-hand limits of $f(x)$ at that point.

\[
(5) \quad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

\[
(6.0) \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx
\]

\[
(6.a) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx
\]

\[
(6.b) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx
\]
11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions
Introduction

- This section concerns three topics:
  - Transition from period $2\pi$ to any period $2L$, for the function $f$, simply by a transformation of scale on the $x$-axis.
  - Simplifications. Only cosine terms if $f$ is even (“Fourier cosine series”). Only sine terms if $f$ is odd (“Fourier sine series”).
  - Expansion of $f$ given for $0 \leq x \leq L$ in two Fourier series, one having only cosine terms and the other only sine terms (“half-range expansions”).
Transition

- Periodic functions in applications may have any period. The notation $p=2L$ for the period is practical because $L$ will be a length of a violin string in Sec. 12.2, of a rod in heat conduction in Sec. 12.5, and so on.
- The transition from period $2\pi$ to be period $p=2L$ is effected by a suitable change of scale, as follows.
- Let $f(x)$ have period $p = 2L$. Then we can introduce a new variable $\nu$ such that $f(x)$, as a function of $\nu$, has period $2\pi$. 
Transition

- If we set

\[(a) \quad x = \frac{p}{2\pi} v \quad \text{so that} \quad (b) \quad v = \frac{2\pi}{p} x = \frac{\pi}{L} x \]

then \( v = \pm \pi \) corresponds to \( x = \pm L \). This means that \( f \), as a function of \( v \), has period \( 2\pi \) and, therefore, a Fourier series of the form

\[f(x) = f\left(\frac{L}{\pi}v\right) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\nu + b_n \sin n\nu)\]

with coefficients obtained from (6) in the last section

\[a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) dv \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) \cos n\nu dv \]

\[b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) \sin n\nu dv\]
Transition

- We could use these formulas directly, but the change to $x$ simplifies calculations. Since

\[ v = \frac{\pi}{L} x \quad \text{we have} \quad dv = \frac{\pi}{L} dx \]

and we integrate over $x$ from $-L$ to $L$. Consequently, we obtain for a function $f(x)$ of period $2L$ the Fourier series

\[ f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \]

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas** ($\pi/L$ in $dx$ cancels $1/\pi$ in (3))
Transition

- Just as in Sec. 11.1, we continue to call (5) with any coefficients a **trigonometric series**. And we can integrate from 0 to $2L$ or over any other interval of length $p=2L$.

\[
\begin{align*}
(0) \quad a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \\
(6) \quad (a) \quad a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \\
(b) \quad b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx
\end{align*}
\]
Example 1

- Find the Fourier series of the function (Fig. 263)

\[ f(x) = \begin{cases} 
0 & \text{if } -2 < x < -1 \\
\frac{k}{2} & \text{if } -1 < x < 1 \\
0 & \text{if } 1 < x < 2 
\end{cases} \]

- Solution. From (6.0) we obtain \( a_0 = \frac{k}{2} \). From (6a) we obtain

\[ a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos \frac{n \pi x}{2} \, dx = \frac{1}{2} \int_{-1}^{1} \frac{k}{2} \cos \frac{n \pi x}{2} \, dx = \frac{2k}{n \pi} \sin \frac{n \pi}{2} \]

Thus \( a_n = 0 \) if \( n \) is even and

\[ a_n = \frac{2k}{n \pi} \quad \text{if } n = 1, 5, 9, \ldots \]

\[ a_n = -\frac{2k}{n \pi} \quad \text{if } n = 3, 7, 11, \ldots \]
Example 1

- From (6b) we find that \( b_n = 0 \) for \( n = 1, 2, \ldots \). Hence the Fourier series is a **Fourier cosine series** (that is, it has no sine terms)

\[
f(x) = \frac{k}{2} + \frac{2k}{\pi} \left( \cos \frac{\pi}{2} x - \frac{1}{3} \cos \frac{3\pi}{2} x + \frac{1}{5} \cos \frac{5\pi}{2} x - + \cdots \right)
\]

\[
a_n = \frac{2k}{n\pi} \quad \text{if } n = 1, 5, 9, \ldots
\]

\[
a_n = -\frac{2k}{n\pi} \quad \text{if } n = 3, 7, 11, \ldots
\]
Example 2

- Find the Fourier series of the function (Fig. 264)

\[ f(x) = \begin{cases} 
-k & \text{if } -2 < x < 0 \\
0 & \text{if } 0 < x < 2 \\
+k & \text{if } 2 < x < 4
\end{cases} \quad p=2L=4, \ L=2 \]

- Solution. Since \( L=2 \), we have in (3) \( v = \frac{\pi x}{2} \) and obtain from (8) in Sec. 11.1 with \( v \) instead of \( x \), that is,

\[ g(v) = \frac{4k}{\pi} \left( \sin v + \frac{1}{3} \sin 3v + \frac{1}{5} \sin 5v + \cdots \right) \]

the present Fourier series

\[ f(x) = \frac{4k}{\pi} \left( \sin \frac{\pi}{2}x + \frac{1}{3} \sin \frac{3\pi}{2}x + \frac{1}{5} \sin \frac{5\pi}{2}x + \cdots \right) \]

Conform this by using (6) and integrating.

\[ \frac{4k}{\pi} (\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots) \]
Example 3

- A sinusoidal voltage $E \sin \omega t$, where $t$ is time, is passed through a half-wave rectifier that clips the negative portion of the wave (Fig. 265). Find the Fourier series of the resulting periodic function

$$u(x) = \begin{cases} 
0 & \text{if } -L < t < 0 \\
E \sin \omega t & \text{if } 0 < t < L 
\end{cases}$$

$\quad p = 2L = \frac{2\pi}{\omega}, \quad L = \frac{\pi}{\omega}$

- Solution. Since $u=0$ when $-L < t < 0$, we obtain from (6.0), with $t$ instead of $x$,

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx$$

(6.0)
Example 3

- From (6a), by using formula (11) in App. A3.1 with $x = \omega t$ and $y = n\omega t$,

$$a_n = \frac{\omega}{2\pi} \int_0^{\pi/\omega} E \sin \omega t \cos n\omega t dt$$

$$= \frac{\omega E}{2\pi} \int_0^{\pi/\omega} [\sin(1 + n)\omega t + \sin(1 - n)\omega t] dt$$

- If $n=1$, the integral on the right is zero, and if $n=2, 3, \ldots$, we readily obtain

$$a_n = \frac{\omega E}{2\pi} \left[ -\frac{\cos(1 + n)\omega t}{(1 + n)\omega} - \frac{\cos(1 - n)\omega t}{(1 - n)\omega} \right]_0^{\pi/\omega}$$

$$= \frac{E}{2\pi} \left( -\frac{\cos(1 + n)\pi + 1}{1 + n} + \frac{-\cos(1 - n)\pi + 1}{1 - n} \right)$$
Example 3

- If \( n \) is odd, this is equal to zero, and for even \( n \) we have
  \[
a_n = \frac{E}{2\pi} \left( \frac{2}{1 + n} + \frac{2}{1 - n} \right) = -\frac{2E}{(n - 1)(n + 1)\pi}
  \]

- In a similar fashion we find from (6b) that \( b_1 = E/2 \) and \( b_n = 0 \) for \( n=2, 3, \ldots \). Consequently,
  \[
u(t) = \frac{E}{\pi} + \frac{E}{2} \sin \omega t - \frac{2E}{\pi} \left( \frac{1}{1 \cdot 3} \cos 2\omega t + \frac{1}{3 \cdot 5} \cos 4\omega t + \cdots \right)
  \]

\[
a_n = \frac{\omega E}{2\pi} \left[ -\frac{\cos(1 + n)\omega t}{(1 + n)\omega} - \frac{\cos(1 - n)\omega t}{(1 - n)\omega} \right]_0^{\pi/\omega}
  = \frac{E}{2\pi} \left( -\frac{\cos(1 + n)\pi + 1}{1 + n} + \frac{\cos(1 - n)\pi + 1}{1 - n} \right)
\]
Simplifications

- If \( f(x) \) is an **even function**, that is, \( f(-x) = f(x) \) (see Fig. 266), its Fourier series (5) reduces to a **Fourier cosine series**

\[
(5*) \quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x
\]

with coefficients (note: integration from 0 to \( L \) only!)

\[
(6*) \quad a_0 = \frac{1}{L} \int_{0}^{L} f(x) \, dx \quad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi}{L} x \, dx
\]
Simplifications

- If \( f(x) \) is an **odd function**, that is, \( f(-x) = -f(x) \) (see Fig. 267), its Fourier series (5) reduces to a **Fourier sine series**

\[
(5^{**}) \quad f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x
\]

with coefficients

\[
(6^{**}) \quad b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx
\]

[Fig. 267. Odd function]
Simplifications

- These formulas follow from (5) and (6) by remembering from calculus that the definite integral gives the net area (=area above the axis minus area below the axis) under the curve of a function between the limits of integration. This implies

\[
\begin{align*}
(7) & \quad (a) \quad \int_{-L}^{L} g(x)\,dx = 2 \int_{0}^{L} g(x)\,dx \quad \text{for even } g \\
& \quad (b) \quad \int_{-L}^{L} h(x)\,dx = 0 \quad \text{for odd } h
\end{align*}
\]

\[
\begin{align*}
(0) & \quad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx \\
(6) & \quad (a) \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx \\
& \quad (b) \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,dx
\end{align*}
\]
Simplifications

- Formula (7b) implies the reduction to the cosine series (even \( f \) makes \( f(x) \sin(n\pi x/L) \) odd since \( \sin \) is odd) and to the sine series (odd \( f \) makes \( f(x) \cos(n\pi x/L) \) odd since \( \cos \) is even).

- Similar, (7a) reduces the integrals in (6*) and (6**) to integrals from 0 to \( L \). These reductions are obvious from the graphs of an even and an odd function.

\[
\begin{align*}
(7) & \quad f_L g(x)dx = 2 f_0 g(x)dx \text{ for even } g \\
& \quad f_L h(x)dx = 0 \text{ for odd } h \\
(0) & \quad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx \\
(6) & \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}dx \\
& \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}dx
\end{align*}
\]
Summary

Even Function of Period $2\pi$. If $f$ is even and $L = \pi$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

with coefficients

$$a_0 = \frac{1}{\pi} \int_{0}^{\pi} f(x) \, dx, \quad a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx, \quad n = 1, 2, \cdots$$

Odd Function of Period $2\pi$. If $f$ is odd and $L = \pi$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

with coefficients

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \cdots.$$
Example 4

- The rectangular wave in Example 1 is even. Hence it follows without calculation that its Fourier series is a Fourier cosine series, the $b_n$ are all zero.
- Similarly, it follows that the Fourier series of the odd function in Example 2 is a Fourier sine series.
- In Example 3 you can see that the Fourier cosine series represents $u(t) = E/\pi - \frac{1}{2}E \sin \omega t$. This is an even function.
Theorem 1

• **Sum and Scalar Multiple**
  
  The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of $f_1$ and $f_2$.

  The Fourier coefficients of $cf$ are $c$ times the corresponding Fourier coefficients of $f$. 
Example 5

- Find the Fourier series of the function (Fig. 268)
  \[ f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi \quad f(x + 2\pi) = f(x) \]

- Solution. We have \( f = f_1 + f_2 \), where \( f_1 = x, f_2 = \pi \). The Fourier coefficients of \( f_2 \) are zero, except for the first one (the constant term), which is \( \pi \). Hence, by Theorem 1, the Fourier coefficients \( a_n, b_n \) are those of \( f_1 \), except for \( a_0 \), which is \( \pi \). Since \( f_1 \) is odd, \( a_n=0 \) for \( n=1, 2, \ldots \), and

\[
b_n = \frac{2}{\pi} \int_0^\pi f_1(x) \sin nx \, dx = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx
\]
Example 5

- Integrating by parts, we obtain

\[ b_n = \frac{2}{\pi} \left[ -\frac{x \cos nx}{n} \right]_0^\pi + \frac{1}{n} \int_0^\pi \cos nx \, dx = \frac{2}{n} \cos n\pi \]

Hence, \( b_1 = 2, b_2 = -\frac{2}{2}, b_3 = \frac{2}{3}, b_4 = -\frac{2}{4}, \ldots \), and the Fourier series of \( f(x) \) is

\[ f(x) = \pi + 2 \left( \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x \ldots \right) \]
Half-Range Expansions

- Half-range expansions are Fourier series. We want to represent \( f(x) \) in Fig. 270.0 by a Fourier series, where \( f(x) \) may be the shape of a distorted violin string or the temperature in a metal bar of length \( L \), for example.

\[
\begin{align*}
&f(x) \\
&x
\end{align*}
\]

- The given function \( f(x) \)

\[
\begin{align*}
&f_1(x) \\
&x \\
&-L \quad L
\end{align*}
\]

(a) \( f(x) \) continued as an even periodic function of period \( 2L \)

\[
\begin{align*}
&f_2(x) \\
&x \\
&-L \quad L
\end{align*}
\]

(b) \( f(x) \) continued as an odd periodic function of period \( 2L \)

*Fig. 270.* Even and odd extensions of period \( 2L \)
Half-Range Expansions

- We would extend $f(x)$ as a function of period $L$ and develop the extended function into a Fourier series. But this series would, in general, contain both cosine and sine terms. We can do better and get simpler series. Indeed, for our given $f$ we can calculate Fourier coefficients from (6*) or form (6**). And we have a choice and can make what seems more practical.

\[
(6*) \quad a_0 = \frac{1}{L} \int_0^L f(x) \, dx \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx
\]

\[
(6**) \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx
\]
Half-Range Expansions

- If we use (6*), we get (5*). This is the **even periodic extension** $f_1$ of $f$ in Fig. 270a. If we use (6**), we get (5**). This is the **odd periodic extension** $f_2$ of $f$ in Fig. 270b.

- Both extensions have period $2L$. This motivates the name **half-range expansions**: $f$ is given only on half the range, that is, on half the interval of periodicity of length $2L$. 

Example 6

- Find the two half-range expansion of the function (Fig. 271)
  \[ f(x) = \begin{cases} 
  \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\
  \frac{2k}{L}(L - x) & \text{if } \frac{L}{2} < x < L 
  \end{cases} \]

- Solution. (a) **Even periodic extension.** From (6*) we obtain
  \[
a_0 = \frac{1}{L} \left[ \frac{2k}{L} \int_0^{L/2} x \, dx + \frac{2k}{L} \int_{L/2}^L (L - x) \, dx \right] = \frac{k}{2}
  \]
  \[
a_n = \frac{2}{L} \left[ \frac{2k}{L} \int_0^{L/2} x \cos \frac{n\pi}{L} \, dx + \frac{2k}{L} \int_{L/2}^L (L - x) \cos \frac{n\pi}{L} \, dx \right]
  \]

We consider \(a_n\). For the first integral we obtain by integration by parts
  \[
  \int_0^{L/2} x \cos \frac{n\pi}{L} \, dx = \frac{Lx}{n\pi} \sin \frac{n\pi}{L} x \bigg|_0^{L/2} - \frac{L}{n\pi} \int_0^{L/2} \sin \frac{n\pi}{L} x \, dx
  \]
  \[
  = \frac{L^2}{2n\pi} \sin \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} (\cos \frac{n\pi}{2} - 1)
  \]
\[ \int uv'dx = uv - \int u'vdx \]

Example 6

- Similarly, for the second integral we obtain
  \[ \int_{L/2}^{L} (L - x) \cos \frac{n\pi}{L} x \, dx = \frac{Lx}{n\pi} (L - x) \sin \frac{n\pi}{L} x \bigg|_{L/2}^{L} + \frac{L}{n\pi} \int_{L/2}^{L} \sin \frac{n\pi}{L} x \, dx \]
  \[ = (0 - \frac{L}{n\pi} (L - \frac{L}{2}) \sin \frac{n\pi}{2}) - \frac{L^2}{n^2\pi^2} (\cos n\pi - \cos \frac{n\pi}{2}) \]

- We insert these two results into the formula for \( a_n \). The sine terms cancel and so does a factor \( L^2 \). This gives
  \[ a_n = \frac{4k}{n^2\pi^2} (2 \cos \frac{n\pi}{2} - \cos n\pi - 1) \]
  Thus, \( a_2 = -16k/(2^2\pi^2) \), \( a_6 = -16k/(6^2\pi^2) \), \( a_{10} = -16k/(10^2\pi^2) \), ... and \( a_n = 0 \) if \( n \neq 2, 6, 10, 14, \ldots \). Hence the first half-range expansion of \( f(x) \) is Fig. (272a)
  \[ f(x) = \frac{k}{2} - \frac{16k}{\pi^2} \left( \frac{1}{2^2} \cos \frac{2\pi}{L} x + \frac{1}{6^2} \cos \frac{6\pi}{L} x + \cdots \right) \]

This Fourier cosine series represents the even periodic extension of the given function \( f(x) \), of period \( 2L \).
Example 6

• (b) Odd periodic extension. Similarly, from (6**) we obtain

\[ b_n = \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2} \]

Hence the other half-range expansion of \( f(x) \) is (Fig. 272b)

\[ f(x) = \frac{8k}{\pi^2} \left( \frac{1}{12} \sin \frac{\pi}{L} x - \frac{1}{3^2} \sin \frac{3\pi}{L} x + \frac{1}{5^2} \sin \frac{5\pi}{L} x - + \cdots \right) \]

The series represents the odd periodic extension of \( f(x) \), of period \( 2L \).
11.3 Forced Oscillations
Forced Oscillations

- Fourier series have important application for both ODEs and PDEs.
- From Sec. 2.8 we know that forced oscillations of a body of mass $m$ on a spring of modulus $k$ are governed by the ODE

\[ my'' + cy' + ky = r(t) \]

where $y = y(t)$ is the displacement from rest, $c$ the damping constant, $k$ the spring constant, and $r(t)$ the external force depending on time $t$. 
Forced Oscillations

- If $r(t)$ is a sine or cosine function and if there is damping ($c>0$), then the steady-state solution is a harmonic oscillation with frequency equal to that of $r(t)$.

- However, if $r(t)$ is not a pure sine or cosine function but is any other periodic function, then the steady-state solution will be a superposition of harmonic oscillations with frequencies equal to that of $r(t)$ and integer multiplies of these frequencies.
Forced Oscillations

- And if one of these frequencies is close to the (practical) resonant frequency of the vibrating system (see Sec. 2.8), then the corresponding oscillation may be the dominant part of the response of the system to the external force.

- This is what the use of Fourier series will show us. Of course, this is quite surprising to an observer unfamiliar with Fourier series, which are highly important in the study of vibrating systems and resonance.
Example 1

In (1), let \( m = 1 \) (g), \( c = 0.05 \) (g/sec), and \( k = 25 \) (g/sec\(^2\)), so that (1) becomes

\[
y'' + 0.05y' + 25y = r(t)
\]

where \( r(t) \) is measured in g \cdot cm/sec\(^2\). Let (Fig. 276)

\[
r(t) =\begin{cases} 
  t + \frac{\pi}{2} & \text{if } -\pi < t < 0 \\
  -t + \frac{\pi}{2} & \text{if } 0 < t < \pi 
\end{cases}
\]

\( r(t + 2\pi) = r(t) \)

Find the steady-state solution \( y(t) \).
Example 1

• Solution. We represent \( r(t) \) by a Fourier series, finding

\[
(3) \quad r(t) = \frac{4}{\pi} \left( \cos t + \frac{1}{3^2} \cos 3t + \frac{1}{5^2} \cos 5t + \cdots \right)
\]

Then we consider the ODE

\[
(4) \quad y'' + 0.05y' + 25y = \frac{4}{n^2\pi} \cos nt \quad (n = 1, 3, \ldots)
\]

whose right side is a single term of the series (3). From Sec. 2.8 we know that the steady-state solution \( y_n(t) \) of (4) is of the form

\[
(5) \quad y_n = A_n \cos nt + B_n \sin nt
\]

By substituting this into (4) we find that

\[
(6) \quad A_n = \frac{4(25 - n^2)}{n^2\pi D_n} \quad B_n = \frac{0.2}{n\pi D_n} \quad D_n = (25 - n^2)^2 + (0.05n)^2
\]
Example 1

- Since the ODE (2) is linear, we may expect the steady-state solution to be

\[ y = y_1 + y_3 + y_5 + \cdots \]  

where \( y_n \) is given by (5) and (6). In fact, this follows readily by substituting (7) into (2) and using the Fourier series of \( r(t) \), provided the termwise differentiation of (7) is permissible.

\[ y_n = A_n \cos nt + B_n \sin nt \]
Example 1

- From (6) we find that the amplitude of (5) is (a factor \( \sqrt{D_n} \) cancels out:)

\[
C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2\pi\sqrt{D_n}}
\]

Values of the first few amplitudes are

\[
C_1 = 0.0531, \quad C_3 = 0.0088, \quad C_5 = 0.2037, \quad C_7 = 0.0011, \quad C_9 = 0.0003
\]

\[
(5) \quad y_n = A_n \cos nt + B_n \sin nt
\]

\[
(6) \quad A_n = \frac{4(25 - n^2)}{n^2\pi D_n} \quad B_n = \frac{0.2}{n\pi D_n} \quad D_n = (25 - n^2)^2 + (0.05n)^2
\]

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Example 1

- Figure 277 shows the input (multiplied by 0.1) and the output. For \( n=5 \) the quantity \( D_n \) is very small, the denominator of \( C_5 \) is small, and \( C_5 \) is so large that \( y_5 \) is the dominating term in (7).

- Hence the output is almost a harmonic oscillation of five times the frequency of the driving force, a little distorted due to the term \( y_1 \), whose amplitude is about 25% of that of \( y_5 \). You could make the situation still more extreme by decreasing the damping constant \( c \).
11.4 Approximation by Trigonometric Polynomials
Approximation

- **Approximation theory**: An area that is concerned with approximating functions by other functions – usually simpler functions

- Let \( f(x) \) be a function on the interval \(-\pi \leq x \leq \pi\) that can be represented on this interval by a Fourier series. Then the \( N \text{th partial sum} \) of the Fourier series

\[
(1) \quad f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)
\]

is an approximation of the given \( f(x) \).
Approximation

- In (1) we choose an arbitrary $N$ and keep it fixed. Then we ask whether (1) is the “best” approximation of $f$ by a trigonometric polynomial of the same degree $N$, that is, by a function of the form
  \[ F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \]  
  \[ (2) \]

- Here, “best” means that the “error” of the approximation is as small as possible.
- We have to define error of such an approximation.
Approximation

- Measure the goodness of agreement between \( f \) and \( F \) on the whole interval \(-\pi \leq x \leq \pi\). This is preferable since the sum \( f \) of a Fourier series may have jumps.

- \( F \) in this figure is a good overall approximation of \( f \), but the maximum of \(|f(x) - F(x)|\) is large. We choose

\[
E = \int_{-\pi}^{\pi} (f - F)^2 dx
\]
Approximation

- This is called the square error of $F$ relative to the function $f$ on the interval $-\pi \leq x \leq \pi$. Clearly $E \geq 0$.
- $N$ being fixed, we want to determine the coefficients in (2) such that $E$ is minimum. Since $(f - F)^2 = f^2 - 2fF + F^2$, we have

$$E = \int_{-\pi}^{\pi} f^2 \, dx - 2 \int_{-\pi}^{\pi} fF \, dx + \int_{-\pi}^{\pi} F^2 \, dx \quad (4)$$

- We square (2), insert it into the last integral in (4), and evaluate the occurring integrals. This gives integrals of $\cos^2 nx$ and $\sin^2 nx$ ($n \geq 1$), which equal $\pi$, and integrals of $\cos nx$, $\sin nx$, and $(\cos nx)(\sin mx)$, which are zero.
Approximation

- Thus

\[ \int_{-\pi}^{\pi} F^2 \, dx = \int_{-\pi}^{\pi} \left[ A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \right]^2 \, dx \]

\[ = \pi\left(2A_0^2 + A_1^2 + \cdots + A_N^2 + B_1^2 + \cdots + B_n^2\right) \]

- We now insert (2) into the integral of \( fF \) in (4). This gives integrals of \( f \cos nx \) as well as \( f \sin nx \), just as in Euler’s formulas, Sec. 11.1, for \( a_n \) and \( b_n \). Hence

\[ \int_{-\pi}^{\pi} fF \, dx = \pi\left(2A_0 a_0 + A_1 a_1 + \cdots + A_N a_N + B_1 b_1 + \cdots + B_N b_N\right) \]
Approximation

- With the expressions, (4) becomes

\[ E = \int_{-\pi}^{\pi} f^2 dx - 2\pi \left[ 2A_0 a_0 + \sum_{n=1}^{N} (A_n a_n + B_n b_n) \right] + \pi \left[ 2A_0^2 + \sum_{n=1}^{N} (A_n^2 + B_n^2) \right] \]

- We now take \( A_n = a_n \) and \( B_n = b_n \) in (2). Then in (5) the second line cancels half of the integral-free expression in the first line. Hence for this choice of the coefficients of \( F \) the square error, call it \( E^* \), is

\[ E^* = \int_{-\pi}^{\pi} f^2 dx - \pi \left[ 2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right] \]
Approximation

- We finally subtract (6) from (5). Then the integrals drop out and we get terms $A_n^2 - 2A_na_n + a_n^2 = (A_n - a_n)^2$ and similar terms $(B_n - b_n)^2$:

$$E - E^* = \pi \left\{ 2(A_0 - a_0)^2 + \sum_{n=1}^{N} [(A_n - a_n)^2 + (B_n - b_n)^2] \right\}$$

- Since the sum of squares of real numbers on the right cannot be negative,

$$E - E^* \geq 0 \quad \Rightarrow \quad E \geq E^*$$

and $E = E^*$ if and only if $A_0 = a_0 \cdots, B_n = b_n$. 


Theorem 1

• Theorem 1: Minimum Square Error

• The square error of $F$ in (2) (with fixed $N$) relative to $f$ on the interval $-\pi \leq x \leq \pi$ is minimum if and only if the coefficients of $F$ in (2) are the Fourier coefficients of $f$. This minimum value $E^*$ is given by (6).

• From (6) we see that $E^*$ cannot increase as $N$ increases, but may decrease. Hence with increasing $N$ the partial sums of the Fourier series of $f$ yield better and better approximations to $f$, considered from the viewpoint of the square error.
Theorem 1

Since $E^* \geq 0$ and (6) holds for every $N$, we obtain from (6) the important **Bessel’s inequality**

\[
2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx
\]

for the Fourier coefficients of any function $f$ for which integral on the right exists.

\[
E^* = \int_{-\pi}^{\pi} f^2 \, dx - \pi \left[ 2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right]
\]
Theorem 1

- It can be shown that for such a function $f$, Parseval’s theorem holds; that is, formula (7) holds with the equality sign, so that it becomes Parseval’s identity

\[
2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx
\]
Example 1

- Compute the minimum square error $E^*$ of $F(x)$ with $N=1, 2, \ldots, 10, 20, \ldots, 100, \text{ and } 1000$ relative to $f(x) = x + \pi$ ($-\pi < x < \pi$) on the interval $-\pi \leq x \leq \pi$

- Solution.

$$F(x) = \pi + 2 \left( \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - + \cdots + \frac{(-1)^{N+1}}{N} \sin N x \right)$$

By Example 3 in Sec. 11.3. From this and (6),

$$E^* = \int_{-\pi}^{\pi} (x + \pi)^2 dx - \pi \left( 2\pi^2 + 4 \sum_{n=1}^{N} \frac{1}{n^2} \right)$$
Example 1

• Numeric values are

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Fig. 269. Partial sums $S_1, S_2, S_3, S_{20}$ in Example 5

Fig. 279. $F$ with $N = 20$ in Example 1
11.5 Sturm-Liouville Problems. Orthogonal Functions
Sturm-Liouville Problem

- Can we replace the trigonometric system by other orthogonal systems (sets of other orthogonal functions)? The answer is yes.

- Consider a second-order ODE of the form

\[
[p(x)y']' + [q(x) + \lambda r(x)]y = 0
\]

on some interval \(a \leq x \leq b\), satisfying equations of the form

\[
\begin{align*}
(2a) & \quad k_1 y + k_2 y' = 0 \quad x = a \\
(2b) & \quad l_1 y + l_2 y' = 0 \quad x = b
\end{align*}
\]

Here \(\lambda\) is a parameter, and \(k_1, k_2, l_1, l_2\) are given real constants.
Sturm-Liouville Problem

- At least one of each constant in each condition (2) must be different from zero.
- Equation (1) is known as a Sturm-Liouville equation. Together with conditions 2(a), 2(b) it is known as the Sturm-Liouville problem.
- A boundary value problem consists of an ODE and given boundary conditions referring to the two boundary points (endpoints) $x=a$ and $x=b$ of a given interval $a \leq x \leq b$
Eigenvalues, Eigenfunctions

- Clearly, \( y \equiv 0 \) is a solution – the trivial solution – of the problem (1), (2) for any \( \lambda \) because (1) is homogeneous and (2) has zeros on the right. This is of no interest.

- We want to find **eigenfunctions** \( y(x) \), that is, solutions of (1) satisfying (2) without being identically zero. We call a number \( \lambda \) for which an eigenfunction exists an **eigenvalue** of the Sturm-Liouville problem (1), (2).

\[
\begin{align*}
(1) & \quad [p(x)y']' + [q(x) + \lambda r(x)]y = 0 \\
(2a) & \quad k_1 y + k_2 y' = 0 \quad x = a \\
(2b) & \quad l_1 y + l_2 y' = 0 \quad x = b
\end{align*}
\]
Example 1

- Find the eigenvalues of eigenfunctions of the Sturm-Liouville problem

\[ y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0 \]

(3) This problem arises, for instance, if an elastic string (a violin string, for example) is stretched a little and fixed at its ends \( x = 0 \) and \( x = \pi \) and then allowed to vibrate. Then \( y(x) \) is the “space function” of the deflection \( u(x, t) \) of the string, assumed in the form \( u(x, t) = y(x)w(t) \), where \( t \) is time.
Example 1

- Solution. From (1) and (2) we see that $p = 1, q = 0, r = 1$ in (1), and $a = 0, b = \pi, k_1 = l_1 = 1, k_2 = l_2 = 0$ in (2). For negative $\lambda = -v^2$ a general solution of the ODE in (3) is $y(x) = c_1 e^{vx} + c_2 e^{-vx}$.

- From the boundary conditions we obtain $c_1 = c_2 = 0$, so that $y \equiv 0$, which is not an eigenfunction. For $\lambda = 0$ the situation is similar. For positive $\lambda = v^2$ a general solution is $y(x) = A \cos vx + B \sin vx$.

\begin{align*}
(1) & \quad [p(x)y']' + [q(x) + \lambda r(x)]y = 0 \\
(2a) & \quad k_1 y + k_2 y' = 0 \quad x = a \\
(2b) & \quad l_1 y + l_2 y' = 0 \quad x = b
\end{align*}
Example 1

- From the first boundary condition we obtain $y(0) = A = 0$. The second boundary condition then yields $y(\pi) = B\sin \nu \pi = 0$, thus $\nu = 0, \pm 1, \pm 2, \ldots$.

- For $\nu = 0$ we have $y \equiv 0$. For $\lambda = \nu^2 = 1, 4, 9, 16, \ldots$, taking $B = 1$, we obtain $y(x) = \sin \nu x$, $\nu = \sqrt{\lambda} = 1, 2, \ldots$.

- Hence the eigenvalues of the problem are $\lambda = \nu^2$, where $\nu = 1, 2, \ldots$, and corresponding eigenfunctions are $y(x) = \sin \nu x$, where $\nu = 1, 2, \ldots$. 

$y(x) = A \cos \nu x + B \sin \nu x$
Eigenvalues, Eigenfunctions

- Note that the solution to this problem is precisely the trigonometric system of the Fourier series considered earlier.

- Under rather general conditions on the functions $p, q, r$ in (1), the Sturm-Liouville problem (1), (2) has infinitely many eigenvalues.

- If $p, q, r, p'$ in (1) are real-valued and continuous on the interval $a \leq x \leq b$ and $r$ is positive throughout that interval, then all the eigenvalues of the Sturm-Liouville problem (1), (2) are real.

\[
\begin{align*}
(1) & \quad [p(x)y']' + [q(x) + \lambda r(x)]y = 0 \\
(2a) & \quad k_1 y + k_2 y' = 0 \quad x = a \\
(2b) & \quad l_1 y + l_2 y' = 0 \quad x = b
\end{align*}
\]
Orthogonal Functions

- Functions $y_1(x), y_2(x), \ldots$ defined on some interval $a \leq x \leq b$ are call **orthogonal** on this interval with respect to the **weight function** $r(x) > 0$ if for all $m$ and all $n$ different from $m$,

\[(y_m, y_n) = \int_a^b r(x)y_m(x)y_n(x)dx = 0\]

$(y_m, y_n)$ is a **standard notation** for this integral. The **norm** $\|y_m\|$ of $y_m$ is defined by

\[\|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x)y_m^2(x)dx}\]

Note that this is the square root of the integral in (4) with $n = m$
Orthogonal Functions

- The functions $y_1, y_2, \ldots$ are called **orthonormal** on $a \leq x \leq b$ if they are orthogonal on this interval and all have norm 1.

- Then we can write (4), (5) jointly by using the **Kronecker symbol** $\delta_{mn}$, namely

  $$ (y_m, y_n) = \int_a^b r(x)y_m(x)y_n(x)dx = \delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} $$

- If $r(x) = 1$, we more briefly call the functions **orthogonal** instead of orthogonal with respect to $r(x) = 1$; similarly for orthogonormality. Then

  $$ (y_m, y_n) = \int_a^b y_m(x)y_n(x)dx = 0, \quad \|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b y_m^2(x)dx} $$
Example 2

- The functions \( y_m(x) = \sin mx, m = 1, 2, \ldots \) form an orthogonal set on the interval \(-\pi \leq x \leq \pi\), because for \( m \neq n \) we obtain by integration:

\[
(y_m, y_n) = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(m-n)x \, dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos(m+n)x \, dx = 0
\]

The norm \( \| y_m \|^2 = \sqrt{(y_m, y_m)} \) equals \( \sqrt{\pi} \) because

\[
\| y_m \|^2 = (y_m, y_m) = \int_{-\pi}^{\pi} \sin^2(mx) \, dx = \pi
\]

Hence the corresponding orthonormal set, obtained by division by the norm, is

\[
\frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\sin 3x}{\sqrt{\pi}}, \ldots
\]
Theorem 1

• Theorem 1. **Orthogonality of Eigenfunctions of Sturm-Liouville Problems**

• Suppose that the functions \( p, q, r, p' \) in the Sturm-Liouville equation (1) are real-valued and continuous and \( r(x) > 0 \) on the interval \( a \leq x \leq b \). Let \( y_m(x) \) and \( y_n(x) \) be eigenfunctions of the Sturm-Liouville problem (1), (2) that correspond to different eigenvalues \( \lambda_m \) and \( \lambda_n \), respectively. Then \( y_m, y_n \) are orthogonal on that interval with respect to the weight function \( r \), that is,

\[
(y_m, y_n) = \int_a^b r(x)y_m(x)y_n(x)dx = 0 \quad (m \neq n)
\]

(6)
Theorem 1

- If \( p(a) = 0 \), then (2a) can be dropped from the problem. If \( p(b) = 0 \), then (2b) can be dropped. [It is then required that \( y \) and \( y' \) remain bounded at such a point, and the problem is called singular, as opposed to a regular problem in which (2) is used.]
- If \( p(a) = p(b) \), then (2) can be replaced by the "periodic boundary conditions"

\[
(7) \quad y(a) = y(b), \quad y'(a) = y'(b)
\]
- The boundary value problem consisting of the Sturm-Liouville equation (1) and the periodic boundary conditions (7) is called a periodic Sturm-Liouville problem.
Example 3

- The ODE in Example 1 is a Sturm-Liouville equation with $p = 1, q = 0, r = 1$. From Theorem 1 it follows that the eigenfunctions $y_m = \sin mx$ ($m = 1, 2, \ldots$) are orthogonal on the interval $0 \leq x \leq \pi$. 

Example 4

- Legendre’s equation \((1 - x^2)y'' - 2xy' + n(n + 1)y = 0\)

may be written
\[
[(1 - x^2)y']' + \lambda y = 0
\]
\[
\lambda = n(n + 1)
\]

Hence, this is a Sturm-Liouville equation (1) with
\(p = 1 - x^2, q = 0, r = 1\). Since \(p(-1) = p(1) = 0\), we
need no boundary conditions, but have a “singular” Sturm-
Liouville problem on the interval \(-1 \leq x \leq 1\). We know
that for \(n = 0, 1, \ldots\), hence \(\lambda = 0, 1 \cdot 2, 2 \cdot 3, \ldots\), the
Legendre polynomials \(P_n(x)\) are solutions of the problem.
Hence these are the eigenfunctions. From Theorem 1 it
follows that they are orthogonal on that interval, that is,
\[
\int_{-1}^{1} P_m(x)P_n(x)dx = 0 \quad (m \neq n)
\]
11.6 Orthogonal Series. Generalized Fourier Series
Orthogonal Series

- Let $y_0, y_1, y_2, \ldots$ be orthogonal with respect to a weight function $r(x)$ on an interval $a \leq x \leq b$, and let $f(x)$ be a function that can be represented by a convergent series

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots$$

- This is called an **orthogonal series**, **orthogonal expansion**, and **generalized Fourier series**.

- If the $y_m$ are the eigenfunctions of a Sturm-Liouville problem, we call (1) an **eigenfunction expansion**.
Given \( f(x) \), we have to determine the coefficients in (1), called the **Fourier constants** of \( f(x) \) with respect to \( y_0, y_1, \ldots \).

Because of the orthogonality, this is simple. We multiply both sides of (1) by \( r(x)y_n(x) \) \((n \text{ fixed})\) and then integrate on both sides from \( a \) to \( b \). We assume that term-by-term integration is permissible. Then we obtain

\[
(f, y_n) = \int_a^b rf y_n dx = \int_a^b r \left( \sum_{m=0}^{\infty} a_m y_m \right) y_n dx = \sum_{m=0}^{\infty} a_m \int_a^b r y_m y_n dx = \sum_{m=0}^{\infty} a_m <y_m, y_n> = \sum_{m=0}^{\infty} a_m (y_m, y_n)
\]
Orthogonal Series

- Because of the orthogonality all the integrals on the right are zero, except when $m = n$. Hence the whole infinite series reduces to the single term

$$a_n (y_n, y_n) = a_n \| y_n \|^2 \quad (f, y_n) = a_n \| y_n \|^2$$

- Assuming that all the functions $y_n$ have nonzero norm, we can divide by $\| y_n \|^2$; writing again $m$ for $n$, to be in agreement with (1), we get the desired formula for the Fourier constants

$$a_m = \frac{(f, y_m)}{\| y_m \|^2} = \frac{1}{\| y_m \|^2} \int_a^b r(x)f(x)y_m(x)dx$$

(2) $\quad (n = 0, 1, \ldots)$
Example 1

- A Fourier-Legendre series is an eigenfunction expansion

\[ f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 P_0 + a_1 P_1(x) + a_2 P_2(x) + \cdots = a_0 + a_1 x + a_2 \left(\frac{3}{2}x^2 - \frac{1}{2}\right) + \cdots \]

in terms of Legendre polynomials. The latter are the eigenfunctions of the Sturm-Liouville problem in Example 4 of Sec. 11.5 on the interval \(-1 \leq x \leq 1\).
Example 1

- We have \( r(x) = 1 \) for Legendre’s equation, and (2) gives

\[
a_m = \frac{2m + 1}{2} \int_{-1}^{1} f(x) P_m(x) \, dx \quad m = 0, 1, \ldots
\]

because the norm is

\[
\|P_m\| = \sqrt{\int_{-1}^{1} P_m(x)^2 \, dx} = \sqrt{\frac{2}{2m + 1}} \quad m = 0, 1, \ldots
\]

as we state without proof.
Example 1

- For instance, let \( f(x) = \sin \pi x \). Then we obtain the coefficients

\[
a_m = \frac{2m + 1}{2} \int_{-1}^{1} (\sin \pi x) P_m(x) \, dx
\]

\[
a_1 = \frac{3}{2} \int_{-1}^{1} x (\sin \pi x) \, dx = \frac{3}{\pi} = 0.95493
\]

Hence the Fourier-Legendre series of \( \sin \pi x \) is

\[
\sin \pi x = 0.95493 P_1(x) - 1.15824 P_3(x) + 0.21929 P_5(x) - 0.01664 P_7(x) + 0.00068 P_9(x) - 0.00002 P_{11}(x) + \cdots
\]

The coefficient of \( P_{13} \) is about \( 3 \cdot 10^{-7} \). The sum of the first three nonzero terms gives a curve that practically coincides with the sine curve.
Mean Square Convergence. Completeness

- A sequence of functions $f_k$ is called \textit{convergent with the limit} $f$ if
  \[ \lim_{k \to \infty} \|f_k - f\| = 0 \]  
  written out by (5) in Sec. 11.5 (where we can drop the square root, as this does not affect the limit)

\[ (12^*) \]

Accordingly, the series (1) converges and represents $f$ if
  \[ \lim_{k \to \infty} \int_a^b r(x) [f_k(x) - f(x)]^2 \, dx = 0 \]

\[ (12) \]

where $s_k$ is the $k$th partial sum of (1)

\[ (13) \]

\[ (14) \]

\[ s_k(x) = \sum_{m=0}^{k} a_m y_m(x) \]
Mean Square Convergence.
Completeness

- An **orthonormal** set \( y_0, y_1, \cdots \) on an interval \( a \leq x \leq b \) is **complete** in a set of functions \( S \) defined on \( a \leq x \leq b \) if we approximate every \( f \) belonging to \( S \) arbitrarily closely in the norm by a linear combination \( a_0 y_0 + a_1 y_1 + \cdots + a_k y_k \), that is, technically, if for every \( \varepsilon > 0 \) we can find constants \( a_0, \cdots, a_k \) (with \( k \) large enough) such that

\[
\| f - (a_0 y_0 + \cdots + a_k y_k) \| < \varepsilon
\]
Mean Square Convergence. Completeness

- Performing the square in (13) and using (14), we first have

\[
\lim_{k \to \infty} \int_a^b r(x) [s_k(x) - f(x)]^2 \, dx = \int_a^b rs_k^2 \, dx - 2 \int_a^b rf \cdot s_k \, dx + \int_a^b rf^2 \, dx
\]

\[
= \int_a^b r \left[ \sum_{m=0}^k a_m y_m \right]^2 \, dx - 2 \sum_{m=0}^k a_m \int_a^b r f y_m \, dx + \int_a^b r f^2 \, dx
\]

The first integral on the right equals \( \sum a_m^2 \) because \( \int r y_m y_l \, dx = 0 \) for \( m \neq l \), and \( \int r y_m^2 \, dx = 1 \). In the second sum on the right, the integral equals \( a_m \), by (2) with \( \|y_m\|^2 = 1 \).
Mean Square Convergence. Completeness

- Hence the first term on the right cancels half of the second term, so that the right side reduces to

\[- \sum_{m=0}^{k} a_m^2 + \int_a^b r f^2 \, dx\]

- This is nonnegative because in the previous formula the integrand on the left is nonnegative (recall that the weight \(r(x)\) is positive!) and so is the integral on the left. This proves the important **Bessel’s inequality**

\[
\sum_{m=0}^{k} a_m^2 \leq \|f\|^2 = \int_a^b r(x)f(x)^2 \, dx \quad (k = 1, 2, \ldots)
\]
Mean Square Convergence. Completeness

- Here we can let $k \to \infty$, because the left sides form a monotone increasing sequence that is bounded by the right side, so that we have convergence by the familiar Theorem 1 in App. A.3.3 Hence

\[
\sum_{m=0}^{\infty} a_m^2 \leq \|f\|^2
\]  

(17)

- Furthermore, if $y_0, y_1, \ldots$ is complete in as set of functions $S$, then (13) holds for every $f$ belonging to $S$. By (13) this implies equality in (16) with $k \to \infty$. 

Mean Square Convergence. Completeness

- Hence in the case of completeness every $f$ in $S$ satisfies the so-called **Parseval equality**

\[
\sum_{m=0}^{\infty} a_m^2 = \|f\|^2 = \int_{a}^{b} r(x)f(x)^2 \, dx
\]

- As a consequence of (18) we prove that in the sense of *completeness* there is no function orthogonal to *every* function of the orthonormal set, with the trivial exception of a function of zero norm.
Theorem 2 Completeness

Let $y_0, y_1, \ldots$ be a complete orthonormal set on $a \leq x \leq b$ in a set of functions $S$. Then if a function $f$ belongs to $S$ and is orthogonal to every $y_m$, it must have norm zero. In particular, if $f$ is continuous, then $f$ must be identically zero.
11.7 Fourier Integral
Fourier Integral

- Manny problems involve functions that are nonperiodic and are of interest on the whole x-axis, we ask what can be done to extend the method of Fourier series to such functions. This idea will lead to “Fourier integrals.”
- We start from a special function $f_L$ of period $2L$ and see what happens to its Fourier series if we let $L \to \infty$. Then we do the same for an arbitrary function $f_L$ of period $2L$. 
Example 1

- Consider the periodic rectangular wave $f_L(x)$ of period $2L > 2$ given by

$$f_L(x) = \begin{cases} 
0 & \text{if } -L < x < -1 \\
1 & \text{if } -1 < x < 1 \\
0 & \text{if } 1 < x < L
\end{cases}$$

The left part of Fig. 280 shows this function from $2L = 4, 8, 16$ as well as the nonperiodic function $f(x)$, which we obtain from $f_L$ if we let $L \to \infty$,

$$f(x) = \lim_{L \to \infty} f_L(x) = \begin{cases} 
1 & \text{if } -1 < x < 1 \\
0 & \text{otherwise}
\end{cases}$$
Example 1

Fig. 280. Waveforms and amplitude spectra in Example 1
Example 1

We now explore what happens to the Fourier coefficients of \( f_L \) as \( L \) increases. Since \( f_L \) is even, \( b_n = 0 \) for all \( n \). For \( a_n \) the Euler formulas (6), Sec. 11.2, give

\[
a_0 = \frac{1}{2L} \int_{-1}^{1} dx = \frac{1}{L}
\]

\[
a_n = \frac{1}{L} \int_{-1}^{1} \cos \frac{n\pi x}{L} \, dx = \frac{2}{L} \int_{0}^{1} \cos \frac{n\pi x}{L} \, dx = \frac{2 \sin(n\pi/L)}{n\pi/L}
\]
Example 1

- This sequence of Fourier coefficients is called the **amplitude spectrum** of $f_L$ because $|a_n|$ is the maximum amplitude of the wave $a_n \cos(n\pi x/L)$. Figure 280 shows this spectrum for the periods $2L = 4, 8, 16$. We see that for increasing $L$ these amplitudes become more and more dense on the positive $w_n$-axis, where $w_n = n\pi/L$. Indeed, for $2L = 4, 8, 16$ we have 1, 3, 7 amplitudes per “half-wave” of the function $(2 \sin w_n)/(Lw_n)$ (dashed in the figure). Hence for $2L = 2^k$ we have $2^{k-1} - 1$ amplitudes per half-wave, so that these amplitudes will eventually be everywhere dense on the positive $w_n$-axis (and will decrease to zero.)
We now consider any periodic function $f_L(x)$ of period $2L$ that can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x)$$

and find out what happens if we let $L \to \infty$. Together with Example 1 the present calculation will suggest that we should expect an integral (instead of a series) involving $\cos wx$ and $\sin wx$ with $w$ no longer restricted to integer multiples $w = w_n = n\pi/L$ of $\pi/L$ but taking all values. We shall also see what form such an integral might have.
From Fourier Series to Fourier Integral

- If we insert $a_n$ and $b_n$ from the Euler formulas (6), Sec. 11.2, and denote the variable of integration by $\nu$, the Fourier series of $f_L(x)$ becomes

$$f_L(x) = \frac{1}{2L} \int_{-L}^{L} f_L(\nu) d\nu$$

$$+ \frac{1}{L} \sum_{n=1}^{\infty} \left[ \cos w_n x \int_{-L}^{L} f_L(\nu) \cos w_n \nu \, d\nu + \sin w_n x \int_{-L}^{L} f_L(\nu) \sin w_n \nu \, d\nu \right]$$

- We now set

$$\Delta w = w_{n+1} - w_n = \frac{(n + 1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$
From Fourier Series to Fourier Integral

- Then \( \frac{1}{L} = \Delta w / \pi \), and we may write the Fourier series in the form

\[
\begin{align*}
  f_L(x) &= \frac{1}{2L} \int_{-L}^{L} f_L(v) dv \\
  &+ \frac{1}{\pi} \sum_{n=1}^{\infty} \left[ \left( \cos w_n x \right) \Delta w \int_{-L}^{L} f_L(v) \cos w_n v dv + \left( \sin w_n x \right) \Delta w \int_{-L}^{L} f_L(v) \sin w_n v dv \right]
\end{align*}
\]

This representation is valid for any fixed \( L \), arbitrarily large, but finite.
From Fourier Series to Fourier Integral

- We now let \( L \rightarrow \infty \) and assume that the resulting nonperiodic function
  \[
  f(x) = \lim_{L \rightarrow \infty} f_L(x)
  \]
  is absolutely integrable on the x-axis; that is, the following (finite!) limits exist:

  \[
  \lim_{g \rightarrow \infty} \int_{-g}^{g} |f(x)| \, dx + \lim_{P \rightarrow \#} \int_{P}^{*} |f(x)| \, dx + \lim_{O \rightarrow \#} \int_{*}^{O} |f(x)| \, dx
  \]

  written \( \int_{-\infty}^{\infty} |f(x)| \, dx \)

\[ (2) \]
From Fourier Series to Fourier Integral

- Then $\frac{1}{L} \to 0$, and the value of the first term on the right side of (1) approaches zero. Also $\Delta w = \frac{\pi}{L} \to 0$ and it seems plausible that the infinite series in (1) becomes an integral from 0 to $\infty$, which represents $f(x)$, namely

$$
(3) \quad f(x) = \frac{1}{\pi} \int_0^\infty \left[ \cos wx \int_{-\infty}^\infty f(v) \cos wvdv + \sin wx \int_{-\infty}^\infty f(v) \sin wvdv \right] dv
$$

$$
(1) \quad f_L(x) = \frac{1}{2L} \int_{-L}^{L} f_L(v) dv + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[ (\cos w_n x) \Delta w \int_{-L}^{L} f_L(v) \cos w_n v dv + (\sin w_n x) \Delta w \int_{-L}^{L} f_L(v) \sin w_n v dv \right]
$$
From Fourier Series to Fourier Integral

- If we introduce the notations

\[
A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv \quad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv
\]

we can write this in the form

\[
f(x) = \frac{1}{\pi} \int_{0}^{\infty} [A(w) \cos wx + B(w) \sin wx] \, dw
\]

This is called a representation of \( f(x) \) by a **Fourier integral**.
Theorem 1 Fourier Integral

- If \( f(x) \) is piecewise continuous (see Sec. 6.1) in every finite interval and has a right-hand derivative and a left-hand derivative at every point (see Sec. 11.1) and if the integral (2) exists, then \( f(x) \) can be represented by a Fourier integral (5) with \( A \) and \( B \) given by (4). At a point where \( f(x) \) is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of \( f(x) \) at that point (see Sec. 11.1).
Example 2

- The main application of Fourier integrals is in solving ODEs and PDEs.
- Find the Fourier integral representation of the function

\[
f(x) = \begin{cases} 
1 & \text{if } |x| < 1 \\
0 & \text{if } |x| > 1
\end{cases}
\]

![Graph of function f(x) with a step function from -1 to 1](image)

**Fig. 281.** Example 2
Example 2

- Solution. From (4) we obtain

\[
A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos vw \, dv = \frac{1}{\pi} \int_{-1}^{1} \cos vw \, dv = \left. \frac{\sin vw}{\pi w} \right|_{-1}^{1} = \frac{2 \sin w}{\pi w}
\]

\[
B(w) = \frac{1}{\pi} \int_{-1}^{1} \sin vw \, dv = 0
\]

and (5) gives the answer

\[
(6) \quad f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\cos wx \sin w}{w} \, dw
\]

The average of the left- and right-hand limits of \( f(x) \) at \( x = 1 \) is equal to \((1 + 0)/2\), that is, \( \frac{1}{2} \).
Example 2

- Furthermore, from (6) and Theorem 1 we obtain (multiply by $\pi/2$)

\[
\int_0^\infty \frac{\cos wx \sin w}{w} \, dw = \begin{cases} 
\frac{\pi}{2} & \text{if } 0 \leq x < 1 \\
\frac{\pi}{4} & \text{if } x = 1 \\
0 & \text{if } x > 1
\end{cases}
\]

(7)

We mention that this integral is called Dirichlet’s discontinuous factor.
Example 2

- The case $x = 0$ is of particular interest. If $x = 0$, then (7) gives

$$(8^*) \quad \int_0^\infty \frac{\sin w}{w} \, dw = \frac{\pi}{2}$$

We see that this integral is the limit of the so-called sine integral

$$(8) \quad Si(u) = \int_0^u \frac{\sin w}{w} \, dw$$

as $u \to \infty$. The graphs of $Si(u)$ and of the integrand are shown in Fig. 282.
Example 2

- In the case of a Fourier series the graphs of the partial sums are approximation curves of the curve of the periodic function represented by the series. Similarly, in the case of the Fourier integral (5), approximations are obtained by replacing $\infty$ by numbers $a$. Hence the integral

$$\frac{2}{\pi} \int_0^a \frac{\cos wx \sin w}{w} \, dw \tag{9}$$

approximates the right side in (6) and therefore $f(x)$.

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos wx \sin w}{w} \, dw \tag{6}$$
Example 2

- Figure 283 shows oscillations near the points of discontinuity of $f(x)$. We might expect that these oscillations disappear as $a$ approaches infinity. But this is not true; with increasing $a$, they are shifted closer to the points $x \pm 1$. This unexpected behavior, which also occurs in connection with Fourier series, is known as the **Gibbs phenomenon**.
Example 2

• Using (11) in App. A3.1, we have

\[ \frac{2}{\pi} \int_0^a \frac{\cos wx \sin w}{w} \, dw = \frac{1}{\pi} \int_0^a \frac{\sin(w + wx)}{w} \, dw + \frac{1}{\pi} \int_0^a \frac{\sin(w - wx)}{w} \, dw \]

• In the first integral on the right we set \( w + wx = t \). Then \( \frac{dw}{w} = \frac{dt}{t} \), and \( 0 \leq w \leq a \) corresponds to \( 0 \leq t \leq (x + 1)a \). In the last integral we set \( w - wx = -t \). Then \( \frac{dw}{w} = \frac{dt}{t} \), and \( 0 \leq w \leq a \) corresponds to \( 0 \leq t \leq (x - 1)a \).
Example 2

Since \( \sin(-t) = -\sin t \), we thus obtain

\[
\frac{2}{\pi} \int_0^a \cos(wx) \frac{\sin w}{w} \, dw = \frac{1}{\pi} \int_0^{(x+1)a} \frac{\sin t}{t} \, dt - \frac{1}{\pi} \int_0^{(x-1)a} \frac{\sin t}{t} \, dt
\]

From this and (8) we see that our integral (9) equals

\[
\frac{1}{\pi} Si(a[x + 1]) - \frac{1}{\pi} Si(a[x - 1])
\]

and the oscillations in Fig. 283 result from those in Fig. 282. The increase of \( a \) amounts to a transformation of the scale on the axis and causes the shift of the oscillations (the waves) toward the points of discontinuity -1 and 1.
Fourier Cosine Integral and Fourier Sine Integral

- If $f$ has a Fourier integral representation and is even, then $B(w) = 0$ in (4). This holds because the integrand of $B(w)$ is odd. Then (5) reduces to a **Fourier cosine integral**

\[
(10) \quad f(x) = \int_0^\infty A(w) \cos wx \, dw \quad A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos vw \, dv
\]

Note the change in $A(w)$: for even $f$ the integrand is even, hence the integral from $-\infty$ to $\infty$ equals twice the integral from 0 to $\infty$, just as in (7a) of Sec. 11.2.

\[
(4) \quad A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos vw \, dv \quad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin vw \, dv
\]

\[
(5) \quad f(x) = \frac{1}{\pi} \int_0^\infty [A(w) \cos wx + B(w) \sin wx] \, dw
\]
Fourier Cosine Integral and Fourier Sine Integral

- Similarly, if \( f \) has a Fourier integral representation and is odd, then \( A(w) = 0 \) in (4). This is true because the integrand of \( A(w) \) is odd. Then (5) becomes a Fourier sine integral

\[
(11) \quad f(x) = \int_0^\infty B(w) \sin wx \, dw \quad B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv
\]

Note the change of \( B(w) \) to an integral from 0 to \( \infty \) because \( B(w) \) is even (odd times odd is even).

\[
(4) \quad A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv \, dv \quad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv \, dv
\]

\[
(5) \quad f(x) = \frac{1}{\pi} \int_0^\infty [A(w) \cos wx + B(w) \sin wx] \, dw
\]
11.8 Fourier Cosine and Sine Transforms
Integral Transform

- An **integral transform** is a transformation in the form of an integral that produces from given function new functions depending on a different variable.

- One is mainly interested in these transforms because they can be used as tools in solving ODEs, PDEs, and integral equations and can often be of help in handling and applying special functions.

- Laplace transform is an example.

\[
F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t) dt
\]
Fourier Cosine Transform

- The Fourier cosine transform concerns **even functions** \( f(x) \). We obtain it from the Fourier cosine integral

\[
f(x) = \int_0^\infty A(w) \cos wx \, dw \quad A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv
\]

Namely, we set \( A(w) = \sqrt{2/\pi} \hat{f}_c (w) \), where \( c \) suggests “cosine”. Then, writing \( v = x \) in the formula for \( A(w) \), we have

\[
(1a) \quad \hat{f}_c (w) = \sqrt{2/\pi} \int_0^\infty f(x) \cos wx \, dx
\]

\[
(1b) \quad f(x) = \sqrt{2/\pi} \int_0^\infty \hat{f}_c (w) \cos wx \, dw
\]
Fourier Cosine Transform

- Formula (1a) gives from \( f(x) \) a new function \( \hat{f}_c(w) \), called the **Fourier cosine transform** of \( f(x) \). Formula (1b) gives us back \( f(x) \) from \( \hat{f}_c(w) \), and we therefore call \( f(x) \) the **inverse Fourier cosine transform** of \( \hat{f}_c(w) \).

- The process of obtaining the transform \( \hat{f}_c \) from a given \( f \) is also called the **Fourier cosine transform** or the **Fourier cosine transform method**.
Fourier Sine Transform

• Similarly, in (11), Sec. 11.7, we set $B(w) = \sqrt{2/\pi} \hat{f}_s(w)$, where $s$ suggests “sine.” Then, writing $\nu = x$, we have from (11), Sec. 11.7, the Fourier sine transform, of $f(x)$ given by

\begin{equation}
\hat{f}_s(w) = \sqrt{2/\pi} \int_0^\infty f(x) \sin wx \, dx \tag{2a}
\end{equation}

and the inverse Fourier sine transform of $\hat{f}_s(w)$, given by

\begin{equation}
f(x) = \sqrt{2/\pi} \int_0^\infty \hat{f}_s(w) \sin wx \, dw \tag{2b}
\end{equation}
Fourier Sine Transform

- The process of obtaining $f_s(w)$ from $f(x)$ is also called the **Fourier sine transform** or the *Fourier sine transform method*.

- Other notations are

$$
\mathcal{F}_c(f) = \hat{f}_c \quad \mathcal{F}_s(f) = \hat{f}_s
$$

and $\mathcal{F}_c^{-1}$ and $\mathcal{F}_s^{-1}$ for the inverses of $\mathcal{F}_c$ and $\mathcal{F}_s$, respectively.
Example 1

- Find the Fourier cosine and Fourier sine transforms of the function

$$f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

- Solution. From the definitions (1a) and (2a) we obtain by integration

$$\hat{f}_c(w) = \sqrt{2/\pi k} \int_0^a \cos wx \, dx = \sqrt{2/\pi k} \left( \frac{\sin aw}{w} \right)$$

$$\hat{f}_s(w) = \sqrt{2/\pi k} \int_0^a \sin wx \, dx = \sqrt{2/\pi k} \left( \frac{1 - \cos aw}{w} \right)$$

This agrees with formulas 1 in the first two tables in Sec. 11.10 (where $k = 1$)
Example 2

- Find $\mathcal{F}_c(e^{-x})$
- Solution. By integration by parts and recursion

$$\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos wx\,dx = \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-x}}{1 + w^2} \right]_{0}^{\infty} = \frac{\sqrt{2/\pi}}{1 + w^2}$$

This agrees with formula 3 in Table I, Sec. 11.10, with $a = 1$. 

Linearity, Transforms of Derivatives

- These transforms have operational properties that permit them to convert differentiations into algebraic operations (just as the Laplace transform does).
- If $f(x)$ is absolutely integrable on the positive $x$-axis and piecewise continuous on every finite interval, then the Fourier cosine and sine transforms of $f$ exists.
- If $f$ and $g$ have Fourier cosine and sine transforms, so does $af + bg$ for any constants $a$ and $b$, and by (1a)

$$
\mathcal{F}_c(af + bg) = \sqrt{2/\pi} \int_0^\infty [af(x) + bg(x)] \cos wx \, dx \\
= a\sqrt{2/\pi} \int_0^\infty f(x) \cos wx \, dx + b\sqrt{2/\pi} \int_0^\infty g(x) \cos wx \, dx
$$
Linearity, Transforms of Derivatives

- The right side is $a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$. Similarly for $\mathcal{F}_s$ by (2). This shows that the Fourier cosine and sine transforms are **linear operations**, 

\[(3a) \quad \mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g)\]

\[(3b) \quad \mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g)\]
Theorem 1 Cosine and Sine Transforms of Derivatives

- Let \( f(x) \) be continuous and absolutely integrable on the x-axis, let \( f'(x) \) be piecewise continuous on every finite interval, and let \( f(x) \to 0 \) as \( x \to \infty \). Then

\[
\mathcal{F}_c(f'(x)) = w\mathcal{F}_s(f(x)) - \sqrt{\frac{2}{\pi}} f(0)
\]

(4a)

\[
\mathcal{F}_s(f'(x)) = -w\mathcal{F}_c(f(x))
\]

(4b)
Theorem 1 Cosine and Sine Transforms of Derivatives

This follows from the definitions and by using integration by parts, namely

\[ F_c(f'(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx \, dx \]

\[ = \sqrt{\frac{2}{\pi}} \left[ f(x) \cos wx \bigg|_0^\infty + w \int_0^\infty f(x) \sin wx \, dx \right] \]

\[ = -\frac{2}{\pi} f(0) + wF_s(f(x)) \]

\[ F_s(f'(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin wx \, dx \]

\[ = \sqrt{\frac{2}{\pi}} \left[ f(x) \sin wx \bigg|_0^\infty - w \int_0^\infty f(x) \cos wx \, dx \right] \]

\[ = 0 - wF_c(f(x)) \]
Linearity, Transforms of Derivatives

- Formula (4a) with \( f' \) instead of \( f \) gives (when \( f', f'' \) satisfy the respective assumptions for \( f, f' \) in Theorem 1)

\[
\mathcal{F}_c(f''(x)) = w\mathcal{F}_s(f'(x)) - \frac{2}{\sqrt{\pi}} f'(0)
\]

hence by (4b)

\[
\text{(5a)} \quad \mathcal{F}_c(f''(x)) = -w^2 \mathcal{F}_c(f(x)) - \frac{2}{\sqrt{\pi}} f'(0)
\]

Similarly,

\[
\text{(5b)} \quad \mathcal{F}_s(f''(x)) = -w^2 \mathcal{F}_s(f(x)) + \frac{2}{\sqrt{\pi}} w f'(0)
\]
Example 3

- Find the Fourier cosine transform $\mathcal{F}_c(e^{-ax})$ of $f(x) = e^{-ax}$, where $a > 0$.
- Solution. By differentiation, $(e^{-ax})'' = a^2 e^{-ax}$; thus $a^2 f(x) = f''(x)$. From this, (5a)., and the linearity (3a)

$$a^2 \mathcal{F}_c(f) = \mathcal{F}_c(f'') = -w^2 \mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}} f'(0) = -w^2 \mathcal{F}_c(f) + a\sqrt{2/\pi}$$

Hence $(a^2 + w^2)\mathcal{F}_c(f) = a\sqrt{2/\pi}$. The answer is

$$\mathcal{F}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \left( \frac{a}{a^2 + w^2} \right) \quad (a > 0)$$
11.9 Fourier Transform. Discrete and Fast Fourier Transforms
Complex Form of the Fourier Integral

- The (real) Fourier integral is [see (4), (5), Sec. 11.7]

\[ f(x) = \int_{0}^{\infty} [A(w) \cos wx + B(w) \sin wx] \, dw \]

where

\[ A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos vw \, dv \quad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin vw \, dv \]

Substituting \( A \) and \( B \) into the integral for \( f \), we have

\[ f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(v) \left[ \cos vw \cos wx + \sin vw \sin wx \right] \, dv \, dw \]
Complex Form of the Fourier Integral

- By the addition formula for the cosine [(6) in App. 3.1] the expression in the brackets [...] equals \( \cos(\omega v - \omega x) \) or, since the cosine is even, \( \cos(\omega x - \omega v) \). We thus obtain

\[
(1^*) \quad f(x) = \frac{1}{\pi} \int_0^\infty \left[ \int_{-\infty}^{\infty} f(\nu) \cos(\omega x - \omega v) \, d\nu \right] \, dw
\]

- The integral in brackets is an even function of \( \omega \), call it \( F(\omega) \), because \( \cos(\omega x - \omega v) \) is an even function of \( \omega \), the function \( f \) does not depend on \( \omega \), and we integrate with respect to \( \nu \) (not \( \omega \)).
Complex Form of the Fourier Integral

- Hence the integral of $F(w)$ from $w = 0$ to $\infty$ is $\frac{1}{2}$ times the integral of $F(w)$ from $-\infty$ to $\infty$. Thus (note the change of the integration limit!)

\[
(1) \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(v) \cos(wx - wv) \, dv \right] \, dw
\]

- We claim that the integral of the form (1) with $\sin$ instead of $\cos$ is zero:

\[
(2) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(v) \sin(wx - wv) \, dv \right] \, dw = 0
\]
Complex Form of the Fourier Integral

\[ (2) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(v) \sin(wx - wv) \, dv \right] \, dw = 0 \]

- This is true since \( \sin(wx - wv) \) is an odd function of \( w \), which makes the integral in brackets an odd function of \( w \), call it \( G(w) \). Hence the integral of \( G(w) \) from \(-\infty\) to \( \infty \) is zero, as claimed.
Complex Form of the Fourier Integral

- We now take the integrand of (1) plus \( i(=\sqrt{-1}) \) times the integrand of (2) and use the Euler formula [(11) in Sec. 2.2]

\[
e^{ix} = \cos x + i \sin x
\]

Taking \( wx - wv \) instead of \( x \) in (3) and multiplying by \( f(v) \) gives

\[
f(v) \cos(wx - wv) + if(v) \sin(wx - wv) = f(v)e^{i(wx-wv)}
\]

Hence the result of adding (1) plus \( i \) times (2), called the complex Fourier integral, is

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(v) e^{iwx-v} dv \right] dw
\]
Fourier Transform and Its Inverse

- Writing the exponential function in (4) as a product of exponential functions, we have

\[
(5) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{i\omega v} dv \right] e^{i\omega x} d\omega
\]

The expression in brackets is a function of \( \omega \), is denoted by \( \hat{f}(\omega) \), and is called the Fourier transform of \( f \); writing \( v = x \), we have

\[
(6) \quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx
\]
Fourier Transform and Its Inverse

- With this, (5) becomes

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)e^{iwx} \, dw
\]

and is called the **inverse Fourier transform** of \( \hat{f}(w) \)

- Another notation for the Fourier transform is \( \hat{f} = \mathcal{F}(f) \) so that \( f = \mathcal{F}^{-1}(\hat{f}) \)
Theorem 1

- Existence of the Fourier Transform
- If $f(x)$ is absolutely integrable on the x-axis and piecewise continuous on every finite interval, then the Fourier transform $\hat{f}(w)$ of $f(x)$ given by (6) exists.
Example 1

- Find the Fourier transform of \( f(x) = 1 \) if \( |x| < 1 \) and \( f(x) = 0 \) otherwise.

- Solution. Using (6) and integrating, we obtain

\[
\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx} \, dx = \frac{1}{\sqrt{2\pi}} \cdot \left. \frac{e^{-iwx}}{-iw} \right|_{-1}^{1} = \frac{1}{-iw\sqrt{2\pi}} (e^{-iw} - e^{iw})
\]

As in (3) we have \( e^{iw} = \cos w + i \sin w \), \( e^{-iw} = \cos w - i \sin w \), and by subtraction \( e^{iw} - e^{-iw} = 2i \sin w \). Substituting this in the previous formula on the right, we see that \( i \) drops out and we obtain the answer

\[
\hat{f}(w) = \sqrt{\frac{\pi}{2}} \frac{\sin w}{w}
\]
Example 2

- Find the Fourier transform $\mathcal{F}(e^{-ax})$ of $f(x) = e^{-ax}$ if $x > 0$ and $f(x) = 0$ if $x < 0$; here $a > 0$.

- Solution. From the definition (6) we obtain by integration

$$\mathcal{F}(e^{-ax}) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-iwx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-(a+iw)x}}{-(a+iw)} \right|_{x=0}^{\infty} = \frac{1}{\sqrt{2\pi}(a+iw)}$$

This proves formula 5 of Table III in Sec. 11.10.
Physical Interpretation: Spectrum

• The nature of the representation (7) of $f(x)$ becomes clear if we think of it as a superposition of sinusoidal oscillations of all possible frequencies, called a spectral representation.

• This name is suggested by optics, where light is such a superposition of colors (frequencies). In (7), the “spectral density” $\hat{f}(w)$ measures the intensity of $f(x)$ in the frequency interval between $w$ and $w + \Delta w$ ($\Delta w$ small, fixed).
Physical Interpretation: Spectrum

- We claim that, in connection with vibrations, the integral \( \int_{-\infty}^{\infty} |\hat{f}(w)|^2 \, dw \) can be interpreted as the **total energy** of the physical system. Hence an integral of \( |\hat{f}(w)|^2 \) from \( a \) to \( b \) gives the contribution of the frequencies \( w \) between \( a \) and \( b \) to the total energy.
Physical Interpretation: Spectrum

To make this plausible, we begin with a mechanical system giving a single frequency, namely, the harmonic oscillator (mass on a spring, Sec. 2.4)

\[ m'y'' + ky = 0 \]

Here we denote time \( t \) by \( x \). Multiplication by \( y' \) gives \( m'y'' + ky'y = 0 \). By integration,

\[ \frac{1}{2}mv^2 + \frac{1}{2}ky^2 = E_0 = \text{const} \]

where \( v = y' \) is the velocity. The first term is the kinetic energy, the second the potential energy, and \( E_0 \) the total energy of the system.
Physical Interpretation: Spectrum

- Now a general solution is (use (3) in Sec. 11.4 with $t = x$)

$$y = a_1 \cos w_0 x + b_1 \sin w_0 x = c_1 e^{i w_0 x} + c_{-1} e^{-i w_0 x}$$

where $c_1 = (a_1 - i b_1)/2$, $c_{-1} = \bar{c}_1 = (a_1 + i b_1)/2$. We write simply $A = c_1 e^{i w_0 x}$, $B = c_{-1} e^{-i w_0 x}$. Then $y = A + B$. By differentiation, $v = y' = A' + B' = i w_0 (A - B)$. Substitution of $v$ and $y$ on the left side of the equation for $E_0$ gives

$$E_0 = \frac{1}{2} m v^2 + \frac{1}{2} k y^2 = \frac{1}{2} m (i w_0)^2 (A - B)^2 + \frac{1}{2} k (A + B)^2$$

$$w_0^2 = k/m$$
Physical Interpretation: Spectrum

- Here $w_0^2 = \frac{k}{m}$, as just stated; hence $mw_0^2 = k$. Also $i^2 = -1$, so that

\[
E_0 = \frac{1}{2} k \left[ -(A - B)^2 + (A + B)^2 \right] = 2kAB
\]
\[
= 2kc_1 e^{i\omega_0 x} c_{-1} e^{-i\omega_0 x} = 2k c_1 c_{-1} = 2k |c_1|^2
\]

Hence the energy is proportional to the square of the amplitude $|c_1|$
Physical Interpretation: Spectrum

- As the next step, if a more complicated system leads to a periodic solution $y = f(x)$ that can be represented by a Fourier series, then instead of the single energy term $|c_1|^2$ we get a series of square $|c_n|^2$ of Fourier coefficients $c_n$ given by (6), Sec. 11.4. In this case we have a “discrete spectrum” (or “point spectrum”) consisting of countably many isolated frequencies (infinitely many, in general), that corresponding $|c_n|^2$ being the contributions to the total energy.
Theorem 2. **Linearity of the Fourier Transform.** The Fourier transform is a **linear operation**; that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants $a$ and $b$, the Fourier transform of $af + bg$ exists, and

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

Proof.

$$\mathcal{F}(af(x) + bg(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)]e^{-iwx} \, dx$$

$$= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx} \, dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{-iwx} \, dx$$

$$= a\mathcal{F}(f(x)) + b\mathcal{F}(g(x))$$
Theorem 3. **Fourier Transform of the Derivative of** \( f(x) \)

Let \( f(x) \) be continuous on the x-axis and \( f(x) \to 0 \) as \( |x| \to \infty \). Furthermore, let \( f'(x) \) be absolutely integrable on the x-axis. Then

\[
\mathcal{F}(f'(x)) = iw\mathcal{F}(f(x))
\]
Linearity. Fourier Transform of Derivatives

- Proof. From the definition of the Fourier transform we have

$$\mathcal{F}(f'(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{-iwx} \, dx$$

Integrating by parts, we obtain

$$\mathcal{F}(f'(x)) = \frac{1}{\sqrt{2\pi}} \left[ f(x)e^{-iwx} \bigg|_{-\infty}^{\infty} - (-iw) \int_{-\infty}^{\infty} f(x)e^{-iwx} \, dx \right]$$

Since $f(x) \to 0$ as $|x| \to \infty$, the desired result follows, namely,

$$\mathcal{F}(f'(x)) = 0 + iw\mathcal{F}(f(x)).$$
Linearity. Fourier Transform of Derivatives

- Two successive applications of (9) give $\mathcal{F}(f''') = i\omega \mathcal{F}(f') = (i\omega)^2 \mathcal{F}(f)$.
- Since $(i\omega)^2 = -\omega^2$, we have for the transform of the second derivative of $f$

$$\mathcal{F}(f''(x)) = -\omega^2 \mathcal{F}(f(x))$$

Similarly for higher derivatives.
Example 3

- Find the Fourier transform of \( xe^{-x^2} \) from Table III, Sec. 11.10

- Solution. We use (9). By formula 9 in Table III

\[
\mathcal{F}(xe^{-x^2}) = \mathcal{F}\left(-\frac{1}{2}(e^{-x^2})'\right)
\]

\[
= -\frac{1}{2} \mathcal{F}\left((e^{-x^2})'\right)
\]

\[
= -\frac{1}{2} iw \mathcal{F}(e^{-x^2})
\]

\[
= -\frac{1}{2} iw \frac{1}{\sqrt{2}} e^{-\frac{w^2}{4}}
\]

\[
= -\frac{iw}{2\sqrt{2}} e^{-w^2/4}
\]
Convolution

- The convolution $f \ast g$ of functions $f$ and $g$ is defined by

$$h(x) = (f \ast g)(x) = \int_{-\infty}^{\infty} f(p)g(x - p)dp = \int_{-\infty}^{\infty} f(x - p)g(p)dp$$

- The purpose is the same as in the case of Laplace transforms (Sec. 6.5): taking the convolution of two functions and then taking the transform of the convolution is the same as multiplying the transforms of these functions (and multiplying them by $\sqrt{2\pi}$).
Convolution

- **Theorem 4. Convolution Theorem.** Suppose that $f(x)$ and $g(x)$ are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then

  \[
  \mathcal{F}(f \ast g) = \sqrt{2\pi} \mathcal{F}(f) \mathcal{F}(g)
  \]

- By taking the inverse Fourier transform on both sides of (12), writing $\hat{f} = \mathcal{F}(f)$ and $\hat{g} = \mathcal{F}(g)$ as before, and noting that $\sqrt{2\pi}$ and $1/\sqrt{2\pi}$ in (12) and (7) cancel each other, we obtain

  \[
  (f \ast g)(x) = \int_{-\infty}^{\infty} \hat{f}(w) \hat{g}(w) e^{iwx} \, dw
  \]
Discrete Fourier Transform (DFT)

- Dealing with sampled values rather than with functions, we can replace the Fourier transform by the so-called **discrete Fourier transform** (DFT).
- Let \( f(x) \) be periodic, for simplicity of period \( 2\pi \). We assume that \( N \) measurements of \( f(x) \) are taken over the interval \( 0 \leq x \leq 2\pi \) at regularly spaced points.

\[
x_k = \frac{2\pi k}{N} \quad k = 0, 1, \ldots, N - 1
\]

- We also say that \( f(x) \) is being **sampled** at these points.
Discrete Fourier Transform (DFT)

- We now want to determine a complex trigonometric polynomial

\[ q(x) = \sum_{n=0}^{N-1} c_n e^{inx_k} \]  

(15)

that interpolates \( f(x) \) at the nodes (14), that is, \( q(x_k) = f(x_k) \), written out, with \( f_k \) denoting \( f(x_k) \),

\[ f_k = f(x_k) = q(x_k) = \sum_{n=0}^{N-1} c_n e^{inx_k} \quad k = 0, 1, \ldots, N - 1 \]  

(16)

Hence we must determine the coefficients \( c_0, \ldots, c_{N-1} \) such that (16) holds.