Chapter 2 Second-Order Linear ODEs

Advanced Engineering Mathematics

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2.1 Homogeneous Linear ODEs of Second Order

Homogeneous Linear ODEs

- A second-order ODE is called **linear** if it can be written

(1) $$y'' + p(x)y' + q(x)y = r(x)$$

and **nonlinear** if it cannot be written in this form.

- If $r(x) \equiv 0$ (that is, $r(x) = 0$ for all x considered; read "$r(x)$ is identically zero"), then (1) reduces to

(2) $$y'' + p(x)y' + q(x)y = 0$$

and is called **homogeneous**. If $r(x) \not\equiv 0$, then (1) is called **nonhomogeneous**.

Homogeneous Linear ODEs

- An example of a nonhomogeneous linear ODE is

$$y'' + 25y = e^{-x} \cos x,$$

and a homogeneous linear ODE is

$$xy'' + y' + xy = 0,$$

written in standard form

$$y'' + \frac{1}{x}y' + y = 0$$

- An example of a nonlinear ODE is

$$y''y + y'^2 = 0.$$

- The functions p and q in (1) and (2) are called the **coefficients** of the ODEs.

Homogeneous Linear ODEs

- **Solutions** are defined similarly as for first-order ODEs in Chap. 1. A function

$$y = h(x)$$

is called a *solution* of a (linear or nonlinear) second-order ODE on some open interval I if h is defined and twice differentiable throughout that interval and is such that the ODE becomes an identity if we replace the unknown y by h, the derivative y' by h', and the second derivative y'' by h''.

Homogeneous Linear ODEs: Superposition Principle

- Linear ODEs have a rich solution structure. For the homogeneous equation the backbone of this structure is the *superposition principle* or *linearity principle*, which says that we can obtain further solutions from given ones by adding them or by multiplying them with any constants. Of course, this is a great advantage of homogeneous linear ODEs.

Example

- The functions $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE $y'' + y = 0$ for all *x*. We verify this by differentiation and substitution.
- We obtain $(\cos x)'' = -\cos x$; hence
$$y'' + y = (\cos x)'' + \cos x = 0$$
- Similarly for $y = \sin x$.
- We multiply $\cos x$ by a constant, for instance, 4.7, and $\sin x$ by, say -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives
$$\begin{aligned}&(4.7\cos x - 2\sin x)'' + (4.7\cos x - 2\sin x)\\ &= -4.7\cos x + 2\sin x + 4.7\cos x - 2\sin x\\ &= 0\end{aligned}$$

Example

- In this example we have obtained from y_1 and y_2 a function of the form

(3) $$y = c_1 y_1 + c_2 y_2$$

- This is called a **linear combination** of y_1 and y_2. In terms of this concept we can now formulate the result suggested by our example, often called the **superposition principle** or **linearity principle**.

Theorem 1

- **Fundamental Theorem for the Homogeneous Linear ODE**
- For a homogeneous linear ODE (2), any linear combination of two solutions on an open interval I is again a solution of (2) on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.
- Don't forget that this theorem holds for homogeneous linear ODEs only!

Example: A Nonhomogeneous Linear ODE

- Verify by substitution that the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions of the nonhomogeneous linear ODEs

$$y'' + y = 1$$

but their sum is not a solution. Neither is, for instance, $2(1 + \cos x)$ or $5(1 + \sin x)$

Example: A Nonlinear ODE

- Verify by substitution that the functions $y = x^2$ and $y = 1$ are solutions of the nonlinear ODE

$$y''y - xy' = 0$$

but their sum is not a solution. Neither is $-x^2$, so you cannot even multiply by -1.

Initial Value Problem

- For a second-order homogeneous linear ODE (2) an **initial value problem** consists of (2) and two **initial conditions**

(4) $y(x_0) = K_0, \qquad y'(x_0) = K_1.$

- These conditions prescribe given values K_0 and K_1 of the solution and its first derivative (the slope of its curve) at the same given $x = x_0$ in the open interval considered.

Initial Value Problem

- The conditions (4) are used to determine the two arbitrary constants c_1 and c_2 in a **general solution**

(5) $$y = c_1 y_1 + c_2 y_2$$

of the ODE; here, y_1 and y_2 are suitable solutions of the ODE.

- This results in a unique solution, passing through the point (x_0, K_0) with K_1 as the tangent direction (the slope) at that point. That solution is called a **particular solution** of the ODE (2).

Example

- Solve the initial value problem

$$y'' + y = 0, y(0) = 3.0, y'(0) = -0.5$$

- Step 1. General solution. The functions $\cos x$ and $\sin x$ are solutions of the ODE and we take

$$y = c_1 \cos x + c_2 \sin x$$

  This will turn out to be a general solution as defined.

- Step 2. Particular solution. We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, since $\cos 0 = 1$ and $\sin 0 = 0$

$$y(0) = c_1 = 3.0 \qquad y'(0) = c_2 = -0.5$$

  This gives as the particular solution

$$y = 3.0 \cos x - 0.5 \sin x$$

Example

- This figure shows that at x=0 it has the value 3.0 and the slope -0.5

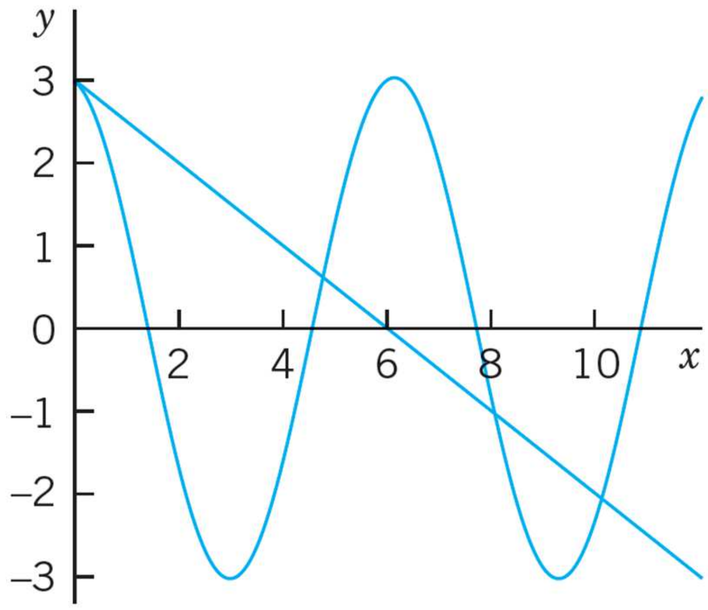

Fig. 29. Particular solution and initial tangent in Example 4

Observation

- Let us take instead two proportional solutions $y_1 = \cos x$ and $y_2 = k\cos x$. Then we can write $y = c_1 y_1 + c_2 y_2$ in the form

$$y = c_1 \cos x + c_2 k \cos x = C \cos x$$

- Hence we are no longer able to satisfy two initial conditions with only one arbitrary constant *C*. Consequently, in defining the concept of a general solution, we must exclude proportionality.

Definition: General Solution, Basis, Particular Solution

- A **general solution** of an ODE (2) on an open interval I is a solution (5) in which y_1 and y_2 are solutions of (2) on I that are not proportional, and c_1 and c_2 are arbitrary constants. These y_1, y_2 are called a **basis** (or a **fundamental system**) of solutions of (2) on I.
- A **particular solution** of (2) on I is obtained if we assign specific values to c_1 and c_2 in (5).

Definition: General Solution, Basis, Particular Solution

- Furthermore, as usual, y_1 and y_2 are called *proportional* on I if for all x on I,

  (6) (a) $y_1 = ky_2$ or (b) $y_2 = ly_1$

  where k and l are numbers, zero or not. (Note that (a) implies (b) if and only if $k \neq 0$).

Definition: General Solution, Basis, Particular Solution

- Two functions y_1 and y_2 are called **linearly independent** on an interval I where they are defined if
  (7) $k_1y_1(x) + k_2y_2(x) = 0$ everywhere on I implies $k_1 = 0$ and $k_2 = 0$.

- And y_1 and y_2 are called **linearly dependent** on I if (7) also holds for some constants k_1, k_2 not both zero. Then, if $k_1 \neq 0$ or $k_2 \neq 0$, we can divide and see that y_1 and y_2 are proportional,

$$y_1 = -\frac{k_2}{k_1}y_2 \qquad y_2 = -\frac{k_1}{k_2}y_1$$

- In contrast, in the case of linear *independence* these functions are not proportional because then we cannot divide in (7).
- A **basis** of solutions of (2) on an open interval I is a pair of linearly independent solutions of (2) on I.

Example

- In the previous example, $\cos x$ and $\sin x$ form a basis of the ODE $y'' + y = 0$ for all *x* because their quotient is $\cot x \neq \text{const}$. Hence $y = c_1 \cos x + c_2 \sin x$ is a general solution. The solution $y = 3.0 \cos x - 0.5 \sin x$ of the initial value problem is a particular solution.

Example

- Verify by substitution that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of the ODE $y'' - y = 0$. Then solve the initial value problem

$$y'' - y = 0, y(0) = 6, y'(0) = -2$$

- $(e^x)'' - e^x = 0$ and $(e^{-x})'' - e^{-x} = 0$ show that e^x and e^{-x} are solutions. They are not proportional, $e^x/e^{-x} \neq \text{const}$ Hence e^x and e^{-x} form a basis for all *x*. We now write down the corresponding general solution and its derivative and equate their values at 0 at the given initial conditions,

$$y = c_1 e^x + c_2 e^{-x} \qquad y' = c_1 e^x - c_2 e^{-x} \qquad \begin{aligned} y(0) &= c_1 + c_2 = 6 \\ y'(0) &= c_1 - c_2 = -2 \end{aligned}$$

c_1=2, c_2=4, so that the answer is $y = 2e^x + 4e^{-x}$

Reduction of Order

- It happens quite often that one solution can be found by inspection or in some other way. Then a second linearly independent solution can be obtained by solving a first-order ODE.
- This is called the method of **reduction of order**.

Example

- Find a basis of solutions of the ODE

$$(x^2 - x)y'' - xy' + y = 0.$$

- Inspection shows that $y_1 = x$ is a solution because $y'_1 = 1$ and $y''_1 = 0$, so that the first term vanishes identically and the second and third terms cancel. The idea of the method is to substitute

$$y_2 = uy_1 = ux, \qquad y_2' = u'x + u, \quad y_2'' = u''x + 2u'$$

into the ODE. This gives

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0.$$

ux and $-xu$ cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0, \qquad (x^2 - x)u'' + (x - 2)u' = 0.$$

Example

$$(x^2 - x)u'' + (x-2)u' = 0$$

- This ODE is of first order in $v = u'$, namely, $(x^2 - x)v' + (x - 2)v = 0$. Separation of variables and integration gives

$$\frac{dv}{v} = -\frac{x-2}{x^2-x}dx = \left(\frac{1}{x-1} - \frac{2}{x}\right)dx$$

$$\ln|v| = \ln|x-1| - 2\ln|x| = \ln\frac{|x-1|}{x^2}$$

部份分式法

$$-\frac{x-2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{1}{x-1} - \frac{2}{x}$$

$$A = -\frac{xB}{x-1} - \frac{x-2}{x-1} \qquad \text{Let } x = 0,\ \text{then } A = -2$$

$$B = -\frac{A(x-1)}{x} - \frac{(x-2)}{x} \qquad \text{Let } x = 1,\ \text{then } B = 1$$

Example

- We need no constant of integration because we want to obtain a particular solution; similarly in the next integration. Taking exponents and integrating again, we obtain

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2} \qquad u = \int v\,dx = \ln|x| + \frac{1}{x}$$

hence

$$y_2 = ux = x\ln|x| + 1$$

- Since $y_1 = x$ and $y_2 = x \ln|x| + 1$ are linearly independent (their quotient is not constant), we have obtained a basis of solutions, valid for all positive x.

Reduction of Order

- In this example we applied reduction of order to a homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = 0$$

Note that we now take the ODE in standard form, with *y*'', not *f*(*x*)*y*''. We assume a solution y_1 of (2), on an open interval *I*, to be known and want to find a basis. For this we need a second linearly independent solution y_2 of (2) on *I*. To get y_2, we substitute

$$y = y_2 = uy_1 \quad y' = y_2' = u'y_1 + uy_1' \quad y'' = y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

into (2).

Reduction of Order

- This gives

(8) $$u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0$$

Collecting terms in u'', u', u, we have

$$u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0$$

Now comes the main point. Since y_1 is a solution of (2), the expression in the last parentheses is zero. Hence *u* is gone, and we are left with an ODE in *u*' and *u*''. We divide this remaining ODE by y_1 and set *u*'=*U*, *u*''=*U*',

$$u'' + u'\frac{2y_1' + py_1}{y_1} = 0 \qquad U' + \left(\frac{2y_1'}{y_1} + p\right)U = 0$$

Reduction of Order

- This is the desired first-order ODE, the reduced ODE. Separation of variables and integration gives

$$\frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + p\right)dx \qquad \ln|U| = -2\ln|y_1| - \int p\,dx$$

- By taking exponents we finally obtain

(9) $$U = \frac{1}{y_1^2} e^{-\int p\,dx}$$

Here $U = u'$, so that $u = \int U\,dx$. Hence the desired second solution is $y_2 = y_1 u = y_1 \int U\,dx$. The quotient $y_2/y_1 = u = \int U\,dx$ cannot be constant (since $U > 0$), so that y_1 and y_2 form a basis of solutions.

2.2 Homogeneous Linear ODEs with Constant Coefficients

Homogeneous Linear ODEs

- We shall now consider second-order homogeneous linear ODEs whose coefficients *a* and *b* are constant,

(1) $$y'' + ay' + by = 0.$$

These equations have important applications in mechanical and electrical vibrations.

- To solve (1), we recall from Sec. 1.5 that the solution of the first-order linear ODE with a constant coefficient *k*

$$y' + ky = 0$$

is an exponential function $y = ce^{-kx}$. This gives us the idea to try as a solution of (1) the function

(2) $$y = e^{\lambda x}.$$

Homogeneous Linear ODEs

- Substituting (2) and its derivatives

$$y' = \lambda e^{\lambda x} \text{ and } y'' = \lambda^2 e^{\lambda x}$$

into our equation (1), we obtain

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

- Hence if λ is a solution of the important **characteristic equation** (or *auxiliary equation*)

(3) $$\lambda^2 + a\lambda + b = 0$$

then the exponential function (2) is a solution of the ODE (1).

(1) $y'' + ay' + by = 0$

(2) $y = e^{\lambda x}$

$$\lambda^2 + a\lambda + b = 0$$

Homogeneous Linear ODEs

- Now from algebra we recall that the roots of this quadratic equation (3) are

(4) $\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b})$ $\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$

(3) and (4) will be basic because our derivation shows that the functions

(5) $y_1 = e^{\lambda_1 x}$ $y_2 = e^{\lambda_2 x}$

are solutions of (1).

Homogeneous Linear ODEs

- From algebra we further know that the quadratic equation (3) may have three kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely,
- **(Case I)** *Two real roots if* $a^2 - 4b > 0$,
- **(Case II)** *A real double root if* $a^2 - 4b = 0$,
- **(Case III)** *Complex conjugate roots if* $a^2 - 4b < 0$.

$$\lambda_1 = \tfrac{1}{2}(-a + \sqrt{a^2 - 4b}) \qquad \lambda_2 = \tfrac{1}{2}(-a - \sqrt{a^2 - 4b})$$

Case 1. Two Distinct Real-Roots

- In this case, a basis of solutions of (1) on any interval is

$$y_1 = e^{\lambda_1 x} \qquad y_2 = e^{\lambda_2 x}$$

because y_1 and y_2 are defined (and real) for all x and their quotient is not constant.

- The corresponding general solution is

(6)
$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y_1 = e^{\lambda_1 x} \qquad y_2 = e^{\lambda_2 x}$$

Example

- We can now solve *y*''-*y*=0 systematically. The characteristic equation is $\lambda^2 - 1 = 0$. Its roots are $\lambda_1 = 1$ and $\lambda_2 = -1$. Hence a basis of solution is e^x and e^{-x} and gives the same general solution as before,

$$y = c_1 e^x + c_2 e^{-x}$$

$$y_1 = e^{\lambda_1 x} \qquad y_2 = e^{\lambda_2 x}$$

Example

- Solve the initial value problem

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

- ***Solution. Step 1. General solution.*** The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0$$

Its roots are

$$\lambda_1 = \tfrac{1}{2}(-1 + \sqrt{9}) = 1 \qquad \lambda_2 = \tfrac{1}{2}(-1 - \sqrt{9}) = -2$$

so that we obtain the general solution

$$y = c_1 e^{x} + c_2 e^{-2x}$$

Example

- ***Step 2. Particular solution.*** Since $y'(x) = c_1e^x - 2c_2e^{-2x}$, we obtain from the general solution and the initial conditions

$$y(0) = c_1 + c_2 = 4,$$

$$y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 1$ and $c_2 = 3$. This gives the *answer* $y = e^x + 3e^{-2x}$.

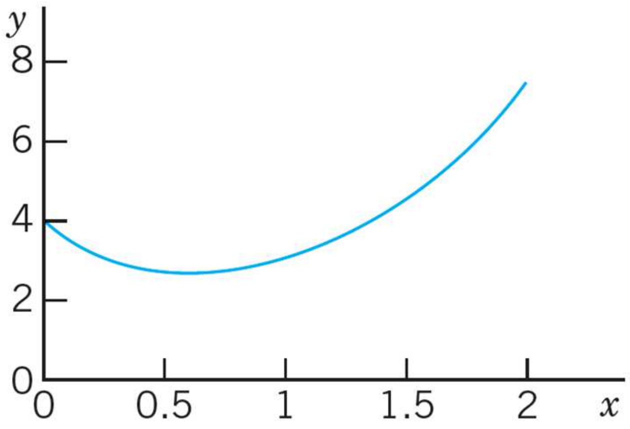

Fig. 30. Solution in Example 2

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}) \qquad \lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$$

Case II. Real Double Root

- If the discriminant $a^2 - 4b$ is zero, we see directly from (4) that we get only one root, $\lambda = \lambda_1 = \lambda_2 = -a/2$, hence only one solution,

$$y_1 = e^{-(a/2)x}$$

- To obtain a second independent solution y_2 (needed for a basis), we use the method of reduction of order discussed in the last section, setting $y_2 = uy_1$. Substituting this and its derivatives $y'_2 = u'y_1 + uy'_1$ and y''_2 into (1), we first have

$$(u''y_1 + 2u'y'_1 + uy''_1) + a(u'y_1 + uy'_1) + buy_1 = 0.$$

$$(1) \quad y'' + ay' + by = 0$$

Case II. Real Double Root

$$y_1 = e^{-(a/2)x}$$

- Collecting terms in u'', u', and u, as in the last section, we obtain

  $u''y_1 + u'(2y'_1 + ay_1) + u(y''_1 + ay'_1 + by_1) = 0.$
- The expression in the last parentheses is zero, since y_1 is a solution of (1). The expression in the first parentheses is zero, too, since

$$2y'_1 = -ae^{-ax/2} = -ay_1.$$

- We are thus left with $u''y_1 = 0$. Hence $u'' = 0$. By two integrations, $u = c_1x + c_2$. To get a second independent solution $y_2 = uy_1$, we can simply choose $c_1 = 1$, $c_2 = 0$ and take $u = x$. Then $y_2 = xy_1$. Since these solutions are not proportional, they form a basis.

Case II. Real Double Root

- Hence in the case of a double root of (3) a basis of solutions of (1) on any interval is

$$e^{-ax/2},\ xe^{-ax/2}.$$

- The corresponding general solution is

(7) $$y = (c_1 + c_2 x)e^{-ax/2}.$$

$$\lambda_1 = \tfrac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right) \qquad \lambda_2 = \tfrac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

Example

- The characteristic equation of the ODE $y'' + 6y' + 9y = 0$ is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$. It has the double root $\lambda = -3$. Hence a basis is e^{-3x} and xe^{-3x}. The corresponding general solution is

$$y = (c_1 + c_2 x)e^{-3x}$$

Example

- Solve the initial value problem

$$y'' + y' + 0.25y = 0, y(0) = 3.0, y'(0) = -3.5$$

- The characteristic equation is $\lambda^2 + \lambda + 0.25 = (\lambda + 0.5)^2 = 0$
  It has the double root $\lambda = -0.5$. This gives the general solution $y = (c_1 + c_2 x)e^{-0.5x}$
- We need its derivative $y' = c_2 e^{-0.5x} - 0.5(c_1 + c_2 x)e^{-0.5x}$
  From this and the initial conditions we obtain
  $y(0) = c_1 = 3.0 \qquad y'(0) = c_2 - 0.5c_1 = -3.5 \qquad c_2 = -2$
  The particular solution is $y = (3 - 2x)e^{-0.5x}$

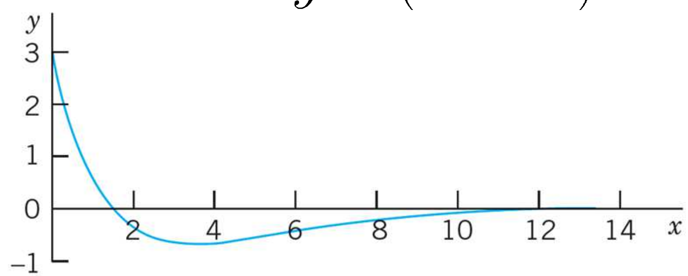

Fig. 31. Solution in Example 4

$$\lambda_1 = \tfrac{1}{2}(-a + \sqrt{a^2 - 4b}) \qquad \lambda_2 = \tfrac{1}{2}(-a - \sqrt{a^2 - 4b})$$

Case III. Complex Roots

- This case occurs if the discriminant $a^2 - 4b$ of the characteristic equation (3) is negative. In this case, the roots of (3) are the complex $\lambda = (-½)a \pm i\omega$ that give the complex solutions of the ODE (1).
- However, we will show that we can obtain a basis of *real* solutions

$$(8) \qquad y_1 = e^{-ax/2} \cos \omega x, \qquad y_2 = e^{-ax/2} \sin \omega x \qquad (\omega > 0)$$

where $\omega^2 = b - (¼)a^2$.

$$(1) \quad y'' + ay' + by = 0$$

Case III. Complex Roots

- It can be verified by substitution that these are solutions in the present case. They form a basis on any interval since their quotient cot ωx is not constant.
- Hence a real general solution in Case III is

(9) $y = e^{-ax/2}(A \cos \omega x + B \sin \omega x) \quad (A, B \text{ arbitrary})$

Example

- Solve the initial value problem

$$y'' + 0.4y' + 9.04y = 0, y(0) = 0, y'(0) = 3$$

- Step 1. General solution. The characteristic equation is $\lambda^2 + 0.4\lambda + 9.04 = 0$. It has the roots $-0.2 \pm 3i$. Hence $\omega = 3$, and a general solution (9) is

$$y = e^{-0.2x}(A\cos 3x + B\sin 3x)$$

- Step 2. Particular solution. The first initial condition gives *y*(0)=*A*=0. The remaining expression $y = Be^{-0.2x}\sin 3x$ We need the derivative

$$y' = B(-0.2e^{-0.2x}\sin 3x + 3e^{-0.2x}\cos 3x)$$

Example

- From this and the second initial condition we obtain *y*'(0)=3*B*=3. Hence *B*=1. Our solution is

$$y = e^{-0.2x} \sin 3x$$

- This figure shows *y* and the curves of $e^{-0.2x}$ and $-e^{-0.2x}$ (dashed), between which the curve of *y* oscillates. Such "damped vibrations" (with *x*=*t* being time) have important mechanical and electrical applications.

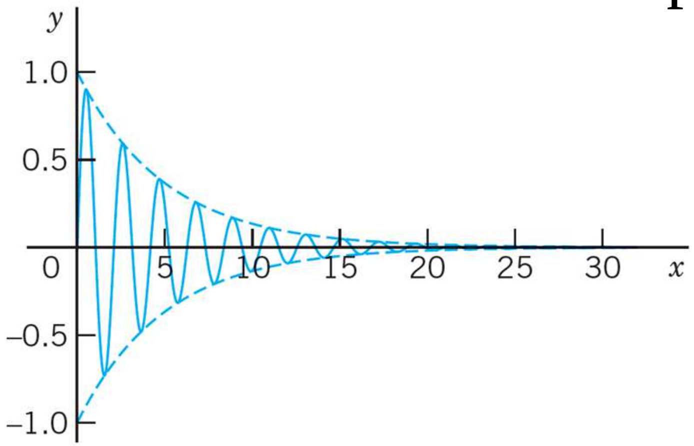

Fig. 32. Solution in Example 5

Summary

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = (-\frac{1}{2})a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$.
III	Complex conjugate $\lambda_1 = (-\frac{1}{2})a + i\omega$ $\lambda_2 = (-\frac{1}{2})a - i\omega$	$e^{-ax/2} \cos \omega x$ $e^{-ax/2} \sin \omega x$	$y = e^{-ax/2} (A \cos \omega x + B \sin \omega x)$

2.3 Differential Operators

Differential Operators

- **Operational calculus** means the technique and application of operators. Here, an **operator** is a transformation that transforms a function into another function. Hence differential calculus involves an operator, the **differential operator** D, which transforms a (differentiable) function into its derivative.
- In operator notation we write $D = d/dx$ and

(1) $$Dy = y' = dy/dx.$$

- Similarly, $D^2y=D(Dy)=y''$.

Differential Operators

- For a homogeneous linear ODE $y'' + ay' + by = 0$ with constant coefficients we can now introduce the **second-order differential operator**

$$L = P(D) = D^2 + aD + bI,$$

where I is the **identity operator** defined by $Iy = y$.

- Then we can write that ODE as

$$(2) \qquad Ly = P(D)y = (D^2 + aD + bI)y = 0.$$

Differential Operators

- P suggests "polynomial." L is a **linear operator**.
- By definition this means that if Ly and Lw exist (this is the case if y and w are twice differentiable), then $L(cy + kw)$ exists for any constants c and k, and

$$L(cy + kw) = cLy + kLw.$$

- *The point of this operational calculus is that* $P(D)$ *can be treated just like an algebraic quantity*. In particular, we can factor it.

Example

- Factor $P(D) = D^2 - 3D - 40I$ and solve $P(D)y = 0$
- $D^2 - 3D - 40I = (D - 8I)(D + 5I)$ because $I^2 = I$
  Now $(D - 8I)y = y' - 8y = 0$ has the solution $y_1 = e^{8x}$
  Similarly, the solution $(D + 5I)y = y' + 5y = 0$ is $y_2 = e^{-5x}$
  This is a basis of $P(D)y = 0$ on any interval.
- From the factorization we obtain the ODE

$$(D-8I)(D+5I)y = (D-8I)(y'+5y) = D(y'+5y) - 8(y'+5y)$$
$$= y'' + 5y' - 8y' - 40y = y'' - 3y' - 40y = 0$$

Verify that this agrees with the result of our method in Sec. 2.2. This is not unexpected because we factored $P(D)$ in the same way s the characteristic polynomial $P(\lambda) = \lambda^2 - 3\lambda - 40$

2.4 Modeling of Free Oscillations of a Mass-Spring System

Setting Up the Model

- We take an ordinary coil spring that resists extension as well as compression. We suspend it vertically from a fixed support and attach a body at its lower end, for instance, an iron ball, as shown in Fig. 33. We let $y = 0$ denote the position of the ball when the system is at rest (Fig. 33b). Furthermore, we choose ***the downward direction as positive***, thus regarding downward forces as *positive* and upward forces as *negative*.

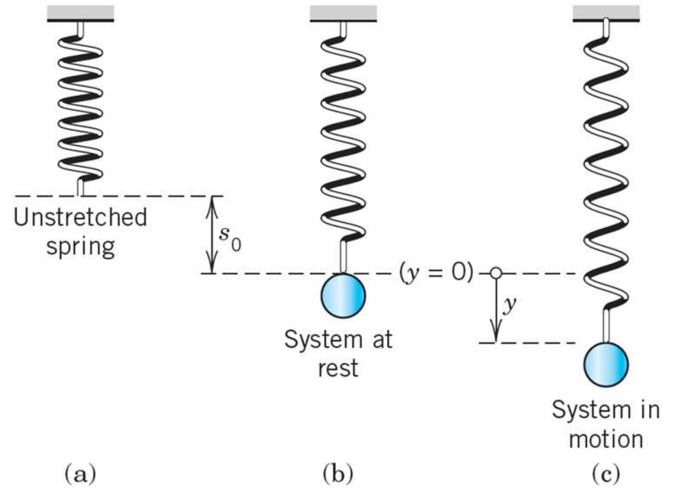

Fig. 33. Mechanical mass–spring system

Setting Up the Model

- We now let the ball move, as follows. We pull it down by an amount $y > 0$ (Fig. 33c). This causes a spring force

(1) $$F_1 = -ky$$ **(Hookes's law)**

proportional to the stretch y, with k (> 0) called the **spring constant**.

- The minus sign indicates that F_1 points upward, against the displacement. It is a *restoring force*: It wants to restore the system, that is, to pull it back to $y = 0$. Stiff springs have large k.

Setting Up the Model

- Note that an additional force $-F_0$ is present in the spring, caused by stretching it in fastening the ball, but F_0 has no effect on the motion because it is in equilibrium with the weight W of the ball,

$$-F_0 = W = mg,$$

where g = 980 cm/sec^2 = 9.8 m/sec^2 = 32.17 ft/sec^2 is the **constant of gravity at the Earth's surface**.

Setting Up the Model

- The motion of our mass–spring system is determined by **Newton's second law**

(2) Mass $\times$ Acceleration = my'' = Force

where $y'' = d^2y/dt^2$ and "Force" is the resultant of all the forces acting on the ball.

ODE of the Undamped System

- Every system has damping. Otherwise it would keep moving forever. But if the damping is small and the motion of the system is considered over a relatively short time, we may disregard damping. Then Newton's law with $F = F_1$ gives the model $my'' = F_1 = -ky$; thus

(3) $$my'' + ky = 0.$$

$$\Rightarrow \quad y'' + \frac{k}{m}y = 0$$

$$\Rightarrow \quad \lambda^2 + \frac{k}{m} = 0 \quad \Rightarrow \quad \lambda = \pm\sqrt{-\frac{k}{m}} = \pm i\omega \qquad \omega = \sqrt{\frac{k}{m}}$$

$$y = e^{-ax/2}(A \cos \omega t + B \sin \omega t)$$

ODE of the Undamped System

- This is a homogeneous linear ODE with constant coefficients. A general solution is obtained as in Sec. 2.2, namely

(4) $y(t) = A \cos \omega_0 t + B \sin \omega_0 t$ $\qquad \omega_0 = \sqrt{\frac{k}{m}}$

- This motion is called a **harmonic oscillation**. Its *frequency* is $f = \omega_0/2\pi$ Hertz (= cycles/sec) because cos and sin in (4) have the period $2\pi/\omega_0$. The frequency f is called the **natural frequency** of the system.

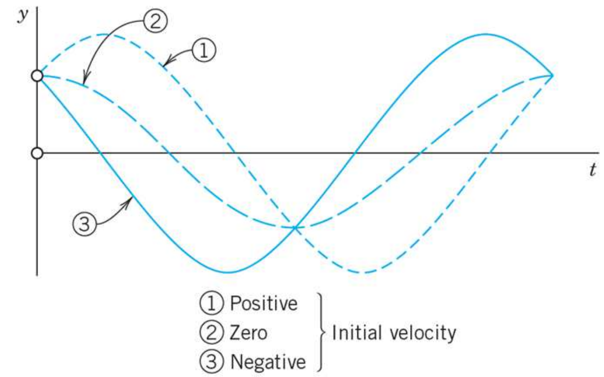

Fig. 34. Typical harmonic oscillations (4) and (4*) with the same $y(0) = A$ and different initial velocities $y'(0) = \omega_0 B$, positive ①, zero ②, negative ③

ODE of the Undamped System

- An alternative representation of (4), which shows the physical characteristics of amplitude and phase shift of (4), is

(4*) $$y(t) = C \cos(\omega_0 t - \delta)$$

with $C = \sqrt{A^2 + B^2}$ and phase angle δ, where $\tan \delta = B/A$.

Example: Harmonic Oscillation of an Undamped Mass-Spring System

- If a mass-spring system with an iron ball of weight W=98 nt can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m, how many cycles per minute will the system execute? What will its motion be if we pull the ball down from rest by 16 cm and let it start with zero initial velocity?

Example: Harmonic Oscillation of an Undamped Mass-Spring System

- Solution: Hooke's law (1) with *W* as the force and 1.09 meter as the stretch gives *W*=1.09k; thus *k*=*W*/1.09=98/1.09=90 [kg/sec^2] = 90 [nt/meter]. The mass is *m*=*W*/*g*=98/9.8=10 [kg]. This gives the frequency $\omega_0/(2\pi) = \sqrt{k/m}/(2\pi) = 3/(2\pi) = 0.48$ [Hz] = 29 [cycles/min].
- From (4) and the initial conditions, $y(0) = A = 0.16$ [meter] and $y'(0) = \omega_0 B = 0$. Hence the motion is

$$y(t) = 0.16 \cos 3t \quad \text{[meter]}$$

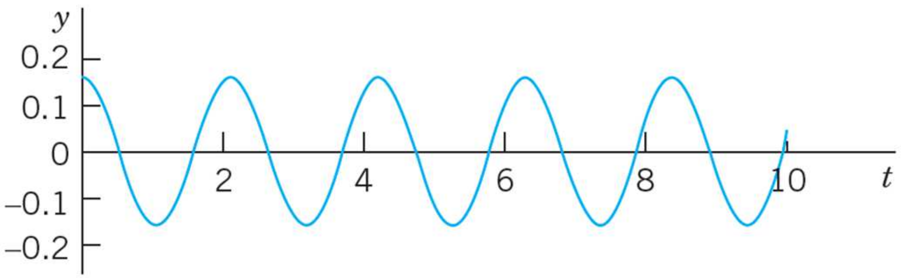

Fig. 35. Harmonic oscillation in Example 1

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

ODE of the Damped System

- To our model $my'' = -ky$ we now add a damping force

$$F_2 = -cy',$$

obtaining $my'' = -ky - cy'$; thus the ODE of the damped mass–spring system is

(5) $$my'' + cy' + ky = 0.$$ (Fig. 36)

- Physically this can be done by connecting the ball to a dashpot; see Fig. 36. We assume this damping force to be proportional to the velocity $y' = dy/dt$. This is generally a good approximation for small velocities.

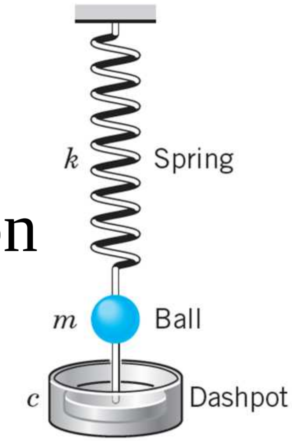

Fig. 36.
Damped system

ODE of the Damped System

- The constant c is called the *damping constant*. Let us show that c is positive. Indeed, the damping force $F_2 = -cy'$ acts *against* the motion; hence for a downward motion we have $y' > 0$, which for positive c makes F negative (an upward force), as it should be.
- Similarly, for an upward motion we have $y' < 0$, which for $c > 0$ makes F_2 positive (a downward force).

$$my'' + cy' + ky = 0$$

ODE of the Damped System

- The ODE (5) is homogeneous linear and has constant coefficients. Hence we can solve it by the method in Sec. 2.2. The characteristic equation is (divide (5) by *m*)

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

- By the usual formula for the roots of a quadratic equation we obtain, as in Sec. 2.2,

$$(6) \quad \lambda_1 = -\alpha + \beta, \lambda_2 = -\alpha - \beta$$

$$\alpha = \frac{c}{2m} \qquad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$$

ODE of the Damped System

- It is now interesting that depending on the amount of damping present—whether a lot of damping, a medium amount of damping, or little damping—three types of motions occur, respectively:

- **Case I.** $c^2 > 4mk$. *Distinct real roots* λ_1, λ_2. **(Overdamping)**

- **Case II.** $c^2 = 4mk$. *A real double root.* **(Critical damping)**

- **Case III.** $c^2 < 4mk$. *Complex conjugate roots.* **(Underdamping)**

$$\lambda_1 = -\alpha + \beta \qquad \lambda_2 = -\alpha - \beta$$

$$\alpha = \frac{c}{2m} \qquad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$$

Case I. Overdamping

- If the damping constant c is so large that $c^2 > 4mk$, then λ_1 and λ_2 are distinct real roots. In this case the corresponding general solution of (5) is

(7) $$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}.$$

- We see that in this case, damping takes out energy so quickly that the body does not oscillate. For $t > 0$ both exponents in (7) are negative because $\alpha > 0$, $\beta > 0$, and $\beta^2 = \alpha^2 - k/m < \alpha^2$. Hence both terms in (7) approach zero as $t \to \infty$
- After a sufficiently long time, the mass will be at rest at the *static equilibrium position* ($y = 0$). Figure 37 shows (7) for some typical initial conditions.

$$\lambda_1 = -\alpha + \beta \qquad \alpha = \frac{c}{2m} \qquad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$$
$$\lambda_2 = -\alpha - \beta$$

Case I. Overdamping

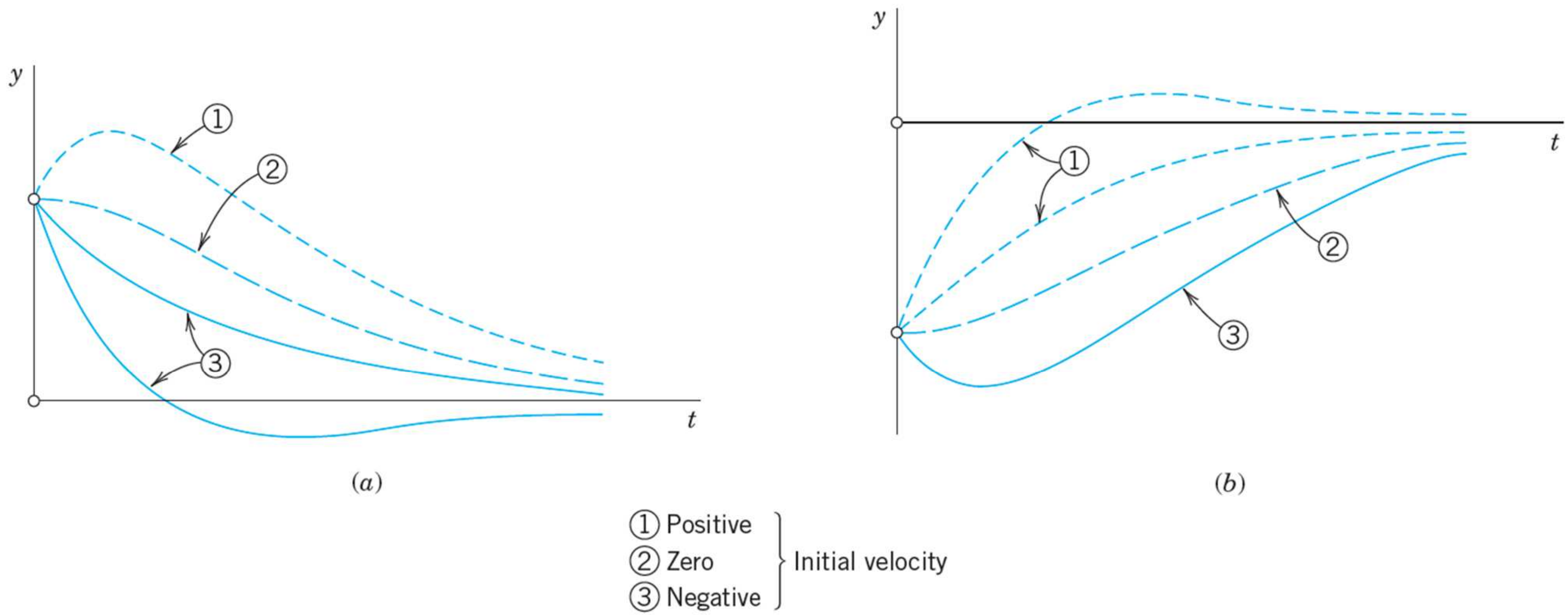

Fig. 37. Typical motions (7) in the overdamped case
(*a*) Positive initial displacement
(*b*) Negative initial displacement

Case II. Critical Damping

- Critical damping is the border case between nonoscillatory motions (Case I) and oscillations (Case III). It occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$. Then the corresponding general solution of (5) is

(8) $$y(t) = (c_1 + c_2 t)e^{-\alpha t}.$$

- This solution can pass through the equilibrium position $y = 0$ at most once because $e^{-\alpha t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero.
- If both $c_1 + c_2$ are positive (or both negative), it has no positive zero, so that y does not pass through 0 at all.

$$\lambda_1 = -\alpha + \beta \qquad \alpha = \frac{c}{2m} \qquad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$$
$$\lambda_2 = -\alpha - \beta$$

Case II. Critical Damping

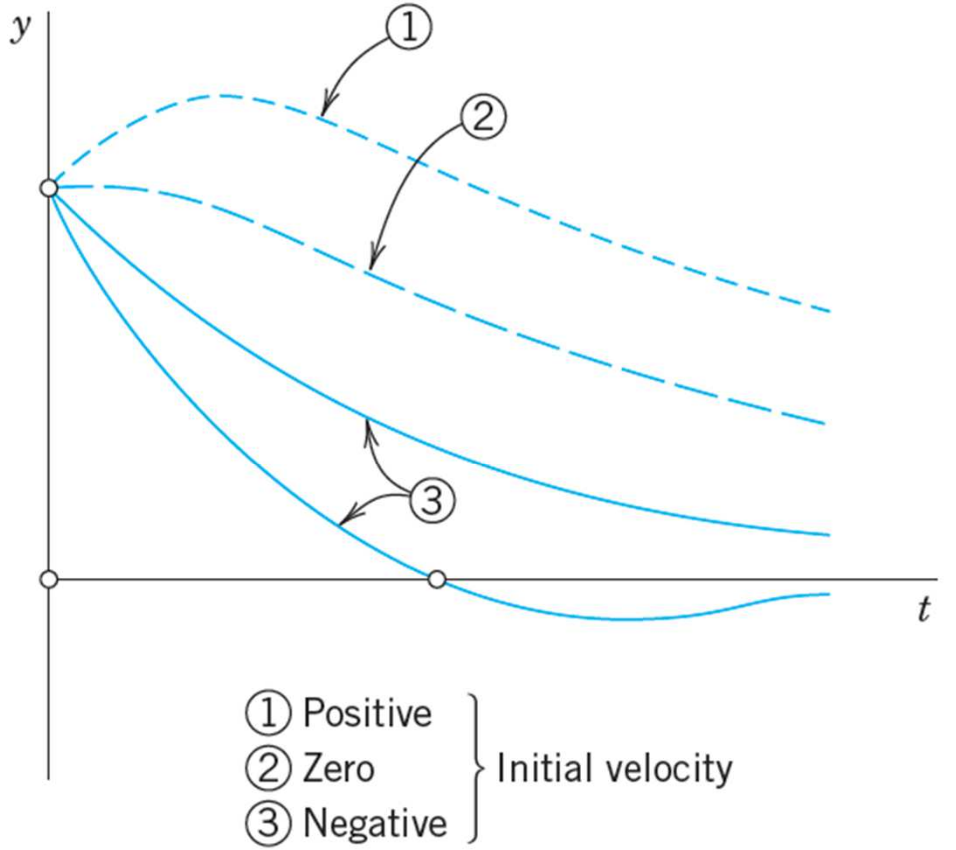

Fig. 38. Critical damping [see (8)]

$$\lambda_1 = -\alpha + \beta \qquad \alpha = \frac{c}{2m} \qquad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$$
$$\lambda_2 = -\alpha - \beta$$

Case III. Underdamping

- It occurs if the damping constant c is so small that $c^2 < 4mk$. Then β in (6) is no longer real but pure imaginary, say,

(9) $$\beta = i\omega^* \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

- The roots of the characteristic equation are now complex conjugates, $\lambda_1 = -\alpha + i\omega^*$, $\lambda_2 = -\alpha - i\omega^*$, with $\alpha = c/(2m)$, as given in (6). Hence the corresponding general solution is

(10) $y(t) = e^{-\alpha t}(A \cos \omega^* t + B \sin \omega^* t) = Ce^{-\alpha t} \cos(\omega^* t - \delta)$

where $C^2 = A^2 + B^2$ and $\tan \delta = B/A$, as in (4*).

Case III. Underdamping

- This represents **damped oscillations**. Their curve lies between the dashed curves $y = Ce^{-\alpha t}$ and $y = -Ce^{-\alpha t}$ in Fig. 39, touching them when $\omega^* t - \delta$ is an integer multiple of π because these are the points at which cos $(\omega^* t - \delta)$ equals 1 or −1.
- The frequency is $\omega^*/(2\pi)$ Hz (hertz, cycles/sec). From (9) we see that the smaller c (>0) is, the larger is ω^* and the more rapid the oscillations become. If c approaches 0, then ω^* approaches $\omega_0 = \sqrt{k/m}$ giving the harmonic oscillation (4), whose frequency $\omega_0/(2\pi)$ is the natural frequency of the system.

Case III. Underdamping

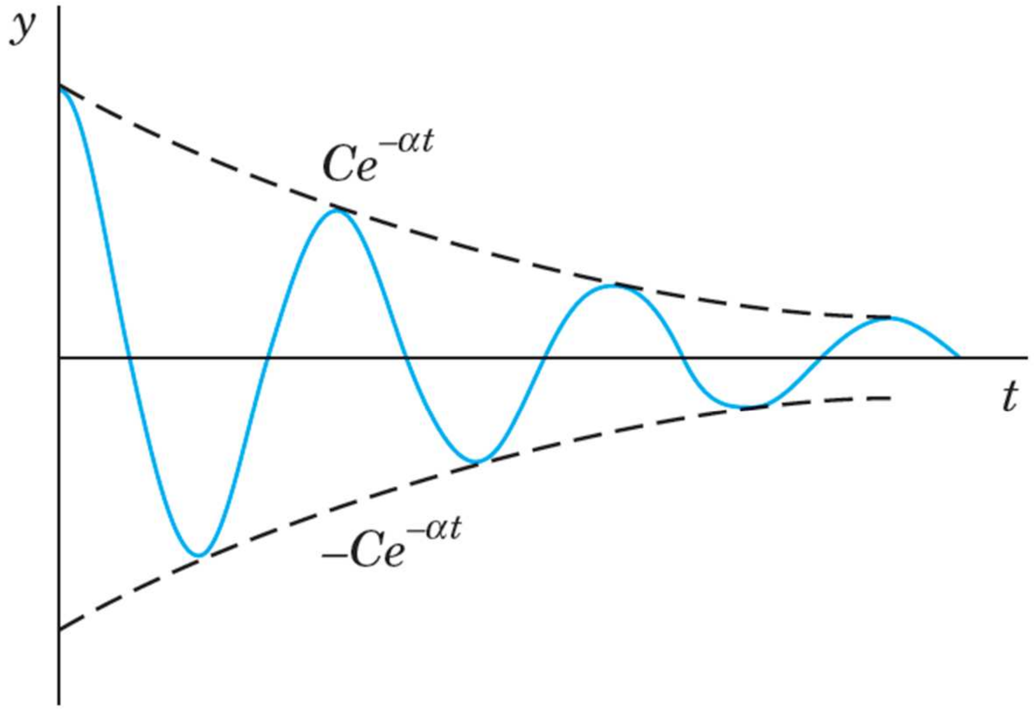

Fig. 39. Damped oscillation in Case III [see (10)]

Example

- How does the motion in the previous example change if we change the damping constant c from one to another of the following three values, with $y(0)=0.16$ and $y'(0)=0$ as before?
  - (I) $c=100$ kg/sec, (II) $c=60$ kg/sec, (III) $c=10$ kg/sec

$my'' + cy' + ky = 0$

Example

- Solution (I). With m=10 and k=90, the model is the initial value problem

  $$10y'' + 100y' + 90y = 0; y(0) = 0.16; y'(0) = 0$$

  The characteristic equation is

  $$10\lambda^2 + 100\lambda + 90 = 10(\lambda + 9)(\lambda + 1) = 0$$

  It has the roots -9 and -1. This gives the general solution $y = c_1 e^{-9t} + c_2 e^{-t}$. We also need $y' = -9c_1 e^{-9t} - c_2 e^{-t}$

  The initial conditions give $c_1 + c_2 = 0.16$, $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is $y = -0.02e^{-9t} + 0.18e^{-t}$

  It approaches 0 as $t \to \infty$

Example

- Solution (II). The model is as before, with c=60. The characteristic equation now has the form $10\lambda^2 + 60\lambda + 90 = 10(\lambda + 3)^2 = 0$. It has the double root -3. Hence the corresponding general solution is $y = (c_1 + c_2 t)e^{-3t}$. We also need $y' = (c_2 - 3c_1 - 3c_2 t)e^{-3t}$ The initial conditions give $y(0)=c_1=0.16$, $y'(0)=c_2-3c_1=0$, $c_2=0.48$. Hence in the critical case the solution is $y = (0.16 + 0.48t)e^{-3t}$
- It is always positive and decreases to 0 in a monotone fashion.

Example

- Solution (III). The model now is $10y'' + 10y' + 90y = 0$ Since c=10 is smaller than the critical c, we shall get oscillations. The characteristic equation $10\lambda^2 + 10\lambda + 90 = 10[(\lambda + \frac{1}{2})^2 + 9 - \frac{1}{4}] = 0$. It has the complex roots

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96i$$

This gives the general solution

$$y = e^{-0.5t}(A\cos 2.96t + B\sin 2.96t)$$

Thus $y(0)=A=0.16$. We also need the derivative

$$y' = e^{-0.5t}(-0.5A\cos 2.96t - 0.5B\sin 2.96t - 2.96A\sin 2.96t + 2.96B\cos 2.96t)$$

Hence $y'(0)=-0.5A+2.96B = 0$, $B=0.027$. This gives the solution

$$y = e^{-0.5t}(0.16\cos 2.96t + 0.027\sin 2.96t) = 0.162e^{-0.5t}\cos(2.96t - 0.17)$$

Example

- We see that these damped oscillations have a smaller frequency than the harmonic oscillations in Example 1 by about 1% (since 2.96 is smaller than 3 by about 1%). Their amplitude goes to zero.

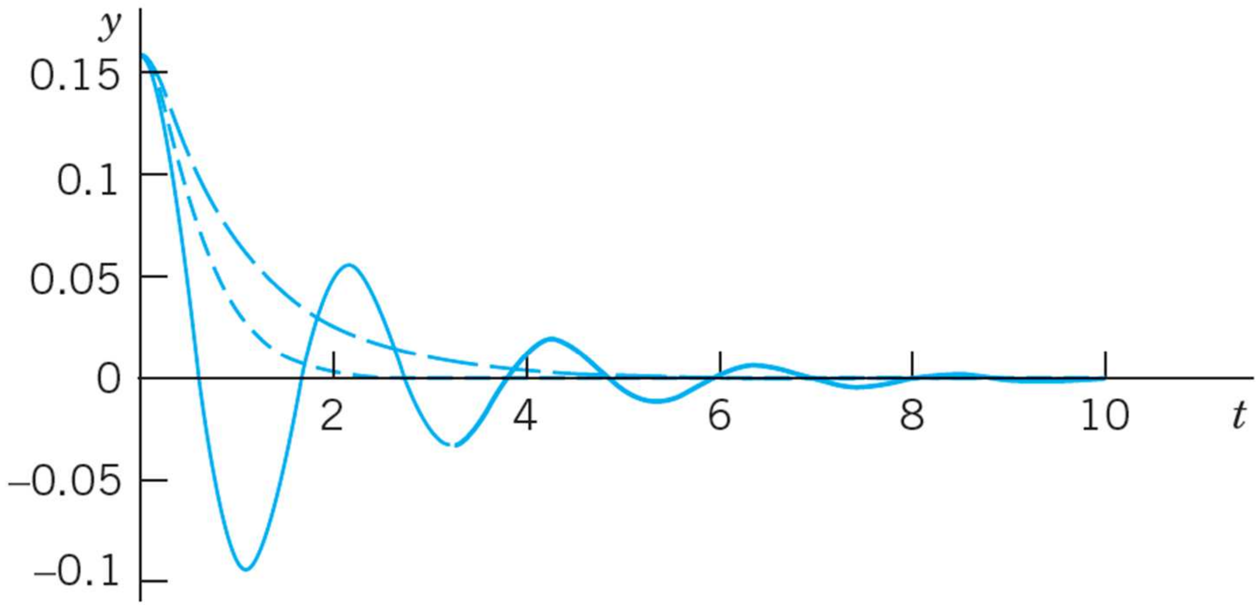

Fig. 40. The three solutions in Example 2

2.5 Euler-Cauchy Equations

Euler-Cauchy Equations

- **Euler–Cauchy equations** are ODEs of the form

(1) $$x^2y'' + axy' + by = 0$$

with given constants a and b and unknown function $y(x)$. We substitute

$$y = x^m, \qquad y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}$$

into (1). This gives

$$x^2m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$$

and we now see that $y = x^m$ was a rather natural choice because we have obtained a common factor x^m. Dropping it, we have the auxiliary equation $m(m-1) + am + b = 0$ or

(2) $$m^2 + (a-1)m + b = 0. \qquad (\textit{Note}: a - 1, \text{ not } a.)$$

$m^2 + (a-1)m + b = 0$

Euler-Cauchy Equations

- Hence $y = x^m$ is a solution of (1) if and only if m is a root of (2). The roots of (2) are

(3) $m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b}$

$m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}$

- **Case I. Real different roots** m_1 and m_2 give two real solutions $y_1(x) = x^{m_1}$ $y_2(x) = x^{m_2}$

  These are linearly independent since their quotient is not constant. Hence they constitute a basis of solutions of (1) for all x for which they are real. The corresponding general solution for all these x is

(4) $y = c_1 x^{m_1} + c_2 x^{m_2}$ (c_1, c_2 arbitrary).

Example

- The Euler-Cauchy equation $x^2y'' + 1.5xy' - 0.5y = 0$ has the auxiliary equation $m^2 + 0.5m - 0.5 = 0$. The roots are 0.5 and -1. Hence a basis of solutions for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution

$$y = c_1\sqrt{x} + \frac{c_2}{x}$$

$$m^2 + (a-1)m + b = 0$$

$$y_1(x) = x^{m_1} \qquad y_2(x) = x^{m_2}$$

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

$m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b} \qquad m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}$

Euler-Cauchy Equations

- **Case II. A real double root** $m_1 = \frac{1}{2}(1-a)$ occurs if and only if $b = \frac{1}{4}(a-1)^2$ because then (2) becomes $[m + \frac{1}{2}(a-1)]^2$ as can be readily verified. Then a solution is $y_1 = x^{(1-a)/2}$, and (1) is of the form

(5) $\quad x^2 y'' + axy' + \frac{1}{4}(1-a)^2 y = 0 \qquad y'' + \frac{a}{x}y' + \frac{(1-a)^2}{4x^2}y = 0$

- A second linearly independent solution can be obtained by the method of reduction of order from Sec. 2.1. Starting from $y_2 = uy_1$, we obtain for u the expression (9) Sec. 2.1, namely

$$u = \int U\,dx \qquad U = \frac{1}{y_1^2}\exp\left(-\int p\,dx\right)$$

(2) $\quad m^2 + (a-1)m + b = 0$

Euler-Cauchy Equations

- From (5) in standard form we see that $p=a/x$ (not ax; this is essential!). Hence

  $$\exp \int (-p\,dx) = \exp(-a \ln x) = \exp(\ln x^{-a}) = 1/x^{a}$$

  Division by $y_1^2 = x^{1-a}$ gives $U=1/x$, so that $u=\ln x$ by integration. Thus, $y_2=uy_1=y_1 \ln x$, and y_1 and y_2 are linearly independent since their quotient is not constant. The general solution corresponding to this basis is

(6) $$y = (c_1 + c_2 \ln x)x^{m} \qquad m = \tfrac{1}{2}(1-a)$$

Example

- The Euler-Cauchy equation $x^2y'' - 5xy' + 9y = 0$ has the auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive x is

$$y = (c_1 + c_2 \ln x)x^3$$

Euler-Cauchy Equations

- **Case III. Complex conjugate roots** are of minor practical importance, and we discuss the derivation of real solutions from complex ones just in terms of a typical example.
- The Euler–Cauchy equation $x^2y'' + 0.6xy' + 16.04y = 0$ has the auxiliary equation $m^2 - 0.4m + 16.04 = 0$. The roots are complex conjugate $m_1 = 0.2 + 4i$ and $m_2 = 0.2 - 4i$, where $i = \sqrt{-1}$. We now use the trick of writing $x = e^{\ln x}$ and obtain

$$x^{m_1} = x^{0.2+4i} = x^{0.2}(e^{\ln x})^{4i} = x^{0.2}e^{(4\ln x)i}$$

$$x^{m_2} = x^{0.2-4i} = x^{0.2}(e^{\ln x})^{-4i} = x^{0.2}e^{-(4\ln x)i}$$

$$e^{it} = \cos t + i \sin t$$

Euler-Cauchy Equations

- Next we apply Euler's formula (11) in Sec. 2.2 with $t = 4 \ln x$ to these two formulas. This gives

$$x^{m_1} = x^{0.2}[\cos(4 \ln x) + i \sin(4 \ln x)]$$
$$x^{m_2} = x^{0.2}[\cos(4 \ln x) - i \sin(4 \ln x)]$$

We now add these two formulas, so that the sine drops out, and divide the result by 2. Then we subtract the second formula from the first, so that the cosine drops out, and divide the result by 2*i*. This yields

$$x^{0.2} \cos(4 \ln x) \qquad x^{0.2} \sin(4 \ln x)$$

respectively.

Euler-Cauchy Equations

- By the superposition principle in Sec. 2.2 these are solutions of the Euler-Cauchy equation (1). Since their quotient $\cot(4 \ln x)$ is not constant, they are linearly independent. Hence they form a basis of solutions, and the corresponding real general solution for all positive x is $y = x^{0.2}[A \cos(4 \ln x) + B \sin(4 \ln x)]$

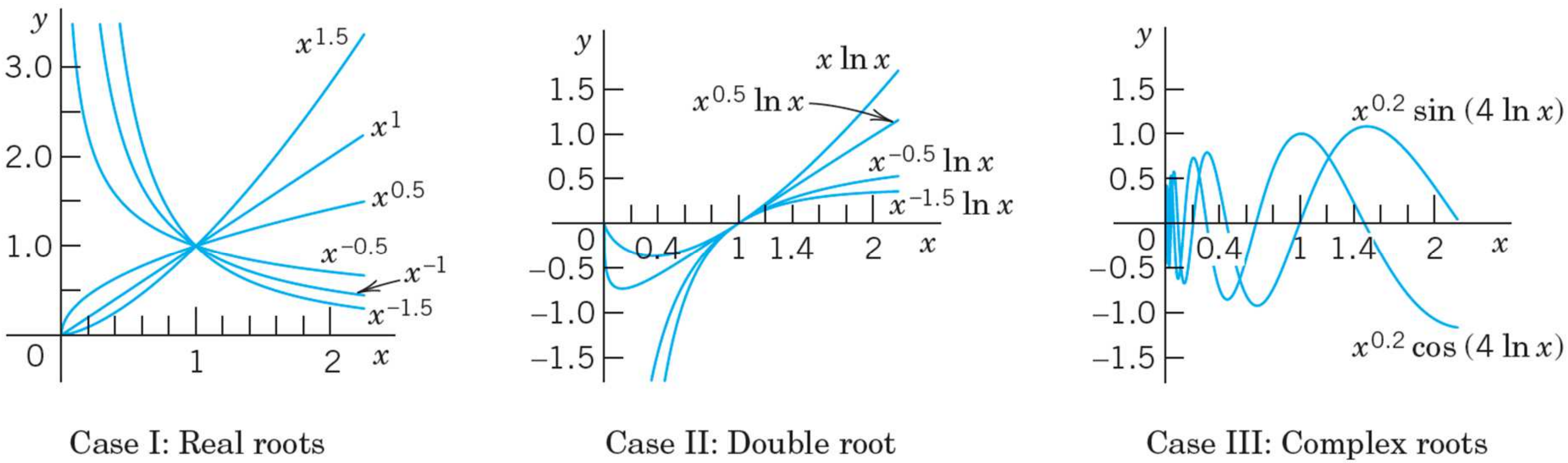

Fig. 48. Euler–Cauchy equations

Example

- Find the electrostatic potential $v=v(r)$ between two concentric spheres of radii r_1=5 cm and r_2=10 cm kept at potentials v_1 = 110 V and v_2=0, respectively.
- Physical information. $v(r)$ is a solution of the Euler-Cauchy equation $rv'' + 2v' = 0$, where $v' = dv/dr$
- Solution. The auxiliary equation $m^2+m=0$. It has the roots 0 and -1. This gives the general solution $v(r) = c_1 + c_2/r$. From the "boundary conditions" (the potentials on the spheres) we obtain

$$v(5) = c_1 + \frac{c_2}{5} = 110 \qquad v(10) = c_1 + \frac{c_2}{10} = 0$$

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

Example

- By subtraction, $c_2/10 = 110, c_2 = 1100.$ From the second equation, $c_1 = -c_2/10 = -110$. So the answer $v(r) = -110 + 1100/r$ V. Figure 49 shows that the potential is not a straight line. As it would be for a potential between two parallel plates. For example, on the sphere of radius 7.5 cm it is not 110/2=55 V, but considerably less.

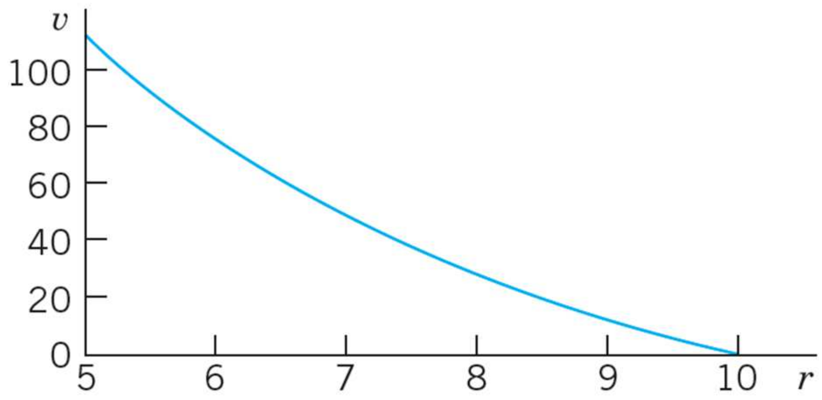

Fig. 49. Potential $v(r)$ in Example 4

2.6 Existence and Uniqueness of Solutions. Wronskian

Existence and Uniqueness of Solutions

- In this section we shall discuss the general theory of homogeneous linear ODEs

(1) $$y'' + p(x)y' + q(x)y = 0$$

with continuous, but otherwise arbitrary, ***variable coefficients*** p and q. This will concern the existence and form of a general solution of (1) as well as the uniqueness of the solution of initial value problems consisting of such an ODE and two initial conditions

(2) $$y(x_0) = K_0, \quad y'(x_0) = K_1$$

with given x_0, K_0, and K_1.

Existence and Uniqueness of Solutions

- The two main results will be Theorem 1, stating that such an initial value problem always has a solution which is unique, and Theorem 4, stating that a general solution

(3) $$y = c_1 y_1 + c_2 y_2 \qquad (c_1, c_2 \text{ arbitrary})$$

includes all solutions. Hence *linear* ODEs with continuous coefficients have no "*singular solutions*" (solutions not obtainable from a general solution).

Theorem 1

- **Existence and Uniqueness Theorem for Initial Value Problems**
- *If* $p(x)$ *and* $q(x)$ *are continuous functions on some open interval* I (see Sec. 1.1) *and* x_0 *is on* I*, then the initial value problem consisting of* (1) *and* (2) *has a unique solution* $y(x)$ *on the interval* I*.*

Linear Independence of Solutions

- A general solution on an open interval I is made up from a **basis** y_1, y_2 on I, that is, from a pair of linearly independent solutions on I. Here we call y_1, y_2 **linearly independent** on I if the equation

(4) $k_1y_1(x) + k_2y_2(x) = 0$ on I implies $k_1 = 0, k_2 = 0$.

- We call y_1, y_2 **linearly dependent** on I if this equation also holds for constants k_1, k_2 not both 0. In this case, and only in this case, y_1 and y_2 are proportional on I, that is (see Sec. 2.1),

(5) (a) $y_1 = ky_2$ or (b) $y_2 = ly_1$ for all on I.

Theorem 2

- **Linear Dependence and Independence of Solutions**
- *Let the ODE* (1) *have continuous coefficients* $p(x)$ *and* $q(x)$ *on an open interval I. Then two solutions* y_1 and y_2 *of* (1) *on I are linearly dependent on I if and only if their* "**Wronskian**"

(6) $$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

is 0 *at some* x_0 *in I. Furthermore, if* $W = 0$ *at an* $x = x_0$ *in I, then* $W = 0$ *on I; hence, if there is an* x_1 *in I at which W is not* 0, *then* y_1, y_2 *are linearly independent on I.*

Theorem 2

- For calculations, the following formulas are often simpler than (6).

(6*) $W(y_1, y_2) =$

(a) $\left(\frac{y_2}{y_1}\right)' y_1^2 \quad (y_1 \neq 0)$ or (b) $\left(\frac{y_1}{y_2}\right)' y_2^2 \quad (y_2 \neq 0)$

(6) $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

Remark

- **Determinants.** Students familiar with second-order determinants may have noticed that

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

- This determinant is called the *Wronski determinant* or, briefly, the **Wronskian**, of two solutions y_1 and y_2 of (1), as has already been mentioned in (6).

Example

- The functions $y_1 = \cos\omega x$ and $y_2 = \sin\omega x$ are solutions of $y'' + \omega^2 y = 0$. Their Wronskian is

$$W(\cos\omega x, \sin\omega x) = \begin{vmatrix} \cos\omega x & \sin\omega x \\ -\omega\sin\omega x & \omega\cos\omega x \end{vmatrix}$$
$$= y_1 y_2' - y_2 y_1'$$
$$= \omega\cos^2\omega x + \omega\sin^2\omega x = \omega$$

Theorem 2 shows that these solutions are linearly independent if and only if $\omega \neq 0$. Of course, we can see this directly from the quotient $y_2/y_1 = \tan\omega x$. For $\omega = 0$ we have $y_2 = 0$, which implies linear dependence.

Example

- A general solution of $y'' - 2y' + y = 0$ on any interval is $y = (c_1 + c_2 x)e^x$. The corresponding Wronskian is not 0, which shows linear independence of e^x and xe^x on any interval. Namely,

$$W(x, e^x) = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = (x+1)e^{2x} - xe^{2x} = e^{2x} \neq 0$$

Theorem 3

- **Existence of a General Solution**
- *If* $p(x)$ *and* $q(x)$ *are continuous on an open interval* I, *then* (1) *has a general solution on* I.

Theorem 4

- **A General Solution Includes All Solutions**
- *If the ODE* (1) *has continuous coefficients* $p(x)$ *and* $q(x)$ *on some open interval I, then every solution* $y = Y(x)$ *of* (1) *on I is of the form*

$$Y(x) = C_1 y_1(x) + C_2 y_2(x) \tag{8}$$

where y_1, y_2 *is any basis of solutions of* (1) *on I and* C_1, C_2 *are suitable constants.*

- *Hence* (1) *does not have* **singular solutions** *(that is, solutions not obtainable from a general solution).*

Proof of Theorem 4

- Let $y=Y(x)$ be any solution of (1) on I. Now, by Theorem 3 the ODE (1) has a general solution

(9) $$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

on I. We have to find suitable values of c_1, c_2 such that $y(x)=Y(x)$ on I. We choose any x_0 in I and show first that we can find values of c_1, c_2 such that we reach agreement at x_0, that is, $y(x_0)=Y(x_0)$ and $y'(x_0)=Y'(x_0)$. Written out in terms of (9), this becomes

(10)
(a) $$c_1 y_1(x_0) + c_2 y_2(x_0) = Y(x_0)$$
(b) $$c_1 y_1'(x_0) + c_2 y_2'(x_0) = Y'(x_0)$$

Proof of Theorem 4

- We determine the unknowns c_1 and c_2. To eliminate c_2, we multiply (10a) by $y_2'(x_0)$ and (10b) by $-y_2(x_0)$ and add the resulting equations. This gives an equation for c_1. Then we multiply (10a) by $-y_1'(x_0)$ and (10b) by $y_1(x_0)$ and add the resulting equations. This gives an equation for c_2. These new equations are as follows, where we take the values of $y_1, y_1', y_2, y_2', Y, Y'$ at x_0

$$c_1(y_1 y_2' - y_2 y_1') = c_1 W(y_1, y_2) = Y y_2' - y2Y'$$

$$c_2(y_1 y_2' - y_2 y_1') = c_2 W(y_1, y_2) = y_1 Y' - Y y_1'$$

Proof of Theorem 4

- Since y_1, y_2 is a basis, the Wronskian W in these equations is not 0, and we can solve for c_1 and c_2. We call the (unique) solution $c_1 = C_1$, $c_2 = C_2$. By substituting it into (9) we obtain from (9) the particular solution $y^*(x) = C_1 y_1(x) + C_2 y_2(x)$
- Now since C_1, C_2 is a solution of (10), we see from (10) that

$$y^*(x_0) = Y(x_0) \qquad y^{*\prime}(x_0) = Y'(x_0)$$

From the uniqueness stated in Theorem 1 this implies that y^* and Y must be equal everywhere on I, and the proof is complete.

2.7 Nonhomogeneous ODEs

Nonhomogeneous ODEs

- Consider the second-order nonhomogeneous linear ODE

(1) $$y'' + p(x)y' + q(x)y = r(x)$$

where $r(x) \neq 0$.We shall see that a "general solution" of (1) is the sum of a general solution of the corresponding homogeneous ODE

(2) $$y'' + p(x)y' + q(x)y = 0$$

and a "particular solution" of (1). These two new terms "general solution of (1)" and "particular solution of (1)" are defined as follows.

General Solution, Particular Solution

- A **general solution** of the nonhomogeneous ODE (1) on an open interval I is a solution of the form

(3) $$y(x) = y_h(x) + y_p(x);$$

here, $y_h = c_1y_1 + c_2y_2$ is a general solution of the homogeneous ODE (2) on I and y_p is any solution of (1) on I containing no arbitrary constants.

- A **particular solution** of (1) on I is a solution obtained from (3) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

Theorem 1

- **Relations of Solutions of (1) to Those of (2)**
- **(a)** *The sum of a solution y of* (1) *on some open interval I and a solution* $\tilde{y}$ *of* (2) *on I is a solution of* (1) *on I. In particular,* (3) *is a solution of* (1) *on I.*
- **(b)** *The difference of two solutions of* (1) *on I is a solution of* (2) *on I.*

(1) $y'' + p(x)y' + q(x)y = r(x)$

(2) $y'' + p(x)y' + q(x)y = 0$

Proof of Theorem 1

- (a) Let $L[y]$ denote the left side of (1). Then for any solutions y of (1) and $\tilde{y}$ of (2) on I.

$$L[y+\tilde{y}] = L[y] + L[\tilde{y}] = r + 0 = r$$

- (b) For any solutions y and y^* on I we have

$$L[y-y^*] = L[y] - L[y^*] = r - r = 0$$

(1) $y'' + p(x)y' + q(x)y = r(x)$

(2) $y'' + p(x)y' + q(x)y = 0$

Theorem 2

- **A General Solution of a Nonhomogeneous ODE Includes All Solutions**
- *If the coefficients* $p(x)$, $q(x)$, *and the function* $r(x)$ *in* (1) *are continuous on some open interval I, then every solution of* (1) *on I is obtained by assigning suitable values to the arbitrary constants* c_1 *and* c_2 *in a general solution* (3) *of* (1) *on I.*

Proof of Theorem 2

- Let y^* be any solution of (1) on I and x_0 any x in I. Let (3) be any general solution of (1) on I. This solution exists. Indeed, $y_h=c_1y_1+c_2y_2$ exists by Theorem 3 in Sec. 2.6 because of the continuity assumption, and y_p exists according to a construction to be shown in Sec. 2.10. Now, by Theorem 1(b) just proved, the difference $Y=y^*-y_p$ is a solution of (2) on I. At x_0 we have

$$Y(x_0) = y^*(x_0) - y_p(x_0) \qquad Y'(x_0) = y^{*\prime}(x_0) - y_p'(x_0)$$

- Theorem 1 in Sec. 2.6 implies that for these conditions, as for any other initial conditions in I, there exists a unique particular solution of (2) obtained by assigning suitable values to c_1,c_2 in y_p. From this and $y^*=Y+y_p$ the statement follows.

Method of Undetermined Coefficients

- *To solve the nonhomogeneous ODE* (1) *or an initial value problem for* (1), *we have to solve the homogeneous ODE* (2) *and find any solution* y_p *of* (1), so that we obtain a general solution (3) of (1).
- How can we find a solution y_p of (1)? One method is the so-called **method of undetermined coefficients**. It is much simpler than another, more general, method (given in Sec. 2.10). Since it applies to models of vibrational systems and electric circuits to be shown in the next two sections, it is frequently used in engineering.

Method of Undetermined Coefficients

- More precisely, the method of undetermined coefficients is suitable for linear ODEs with ***constant coefficients a and b***

(4) $$y'' + ay' + by = r(x)$$

when $r(x)$ is an exponential function, a power of x, a cosine or sine, or sums or products of such functions. These functions have derivatives similar to $r(x)$ itself. This gives the idea.

- We choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE. Table 2.1 on p. 82 shows the choice of y_p for practically important forms of $r(x)$.

Choice Rules for the Method of Undetermined Coefficients

- **(a) Basic Rule.** *If* $r(x)$ *in* (4) *is one of the functions in the first column in Table* 2.1, *choose* y_p *in the same line and determine its undetermined coefficients by substituting* y_p *and its derivatives into* (4).

- **(b) Modification Rule.** *If a term in your choice for* y_p *happens to be a solution of the homogeneous ODE corresponding to* (4), *multiply this term by* x (*or by* x^2 *if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE*).

- **(c) Sum Rule.** *If* $r(x)$ *is a sum of functions in the first column of Table* 2.1, *choose for* y_p *the sum of the functions in the corresponding lines of the second column.*

Choice Rules for the Method of Undetermined Coefficients

- The Basic Rule applies when $r(x)$ is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of (1) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of (1) with $r = r_1 + r_2$.
- The method is self-correcting. A false choice for y_p or one with too few terms will lead to a contradiction. A choice with too many terms will give a correct result, with superfluous coefficients coming out zero.

Choice Rules for the Method of Undetermined Coefficients

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

Example: Application of the Basic Rule (a)

- Solve the initial value problem

(5) $$y'' + y = 0.001x^2, y(0) = 0, y'(0) = 1.5$$

- Step 1. General solution of the homogeneous ODE. The ODE $y'' + y = 0$ has the general solution

$$y_h = A\cos x + B\sin x$$

- Step 2. Solution y_p of the nonhomogeneous ODE. We first try $y_p = Kx^2$. Then $y_p'' = 2K$. By substitution, $2K + Kx^2 = 0.001x^2$. For this to hold all x, the coefficient of each power of x (x^2 and x^0) must be the same on both sides; thus K=0.001 and 2K=0, a contradiction.

Example: Application of the Basic Rule (a)

- The second line in Table 2.1 suggest the choice

$$y_p = K_2 x^2 + K_1 x + K_0$$

$$y_p'' + y_p = 2K_2 + K_2 x^2 + K_1 x + K_0 = 0.001x^2$$

  Equating the coefficients of x^2, x, x^0 on both sides, we have K_2=0.001, K_1=0, $2K_2+K_0$=0. Hence K_0=-2K_2=-0.002. This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002$$

- Step 3. Solution of the initial value problem. Setting x=0 and using the first initial condition gives $y(0)=A-0.002=0$, hence A=0.002.

Example: Application of the Basic Rule (a)

- By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A\sin x + B\cos x + 0.002x$$

$$y'(0) = B = 1.5$$

This gives the answer

$$y = 0.002\cos x + 1.5\sin x + 0.001x^2 - 0.002$$

Figure 50 shows *y* as well as the quadratic parabola y_p about which *y* is oscillating, practically like a sine curve since the cosine term is smaller by a factor of about 1/1000.

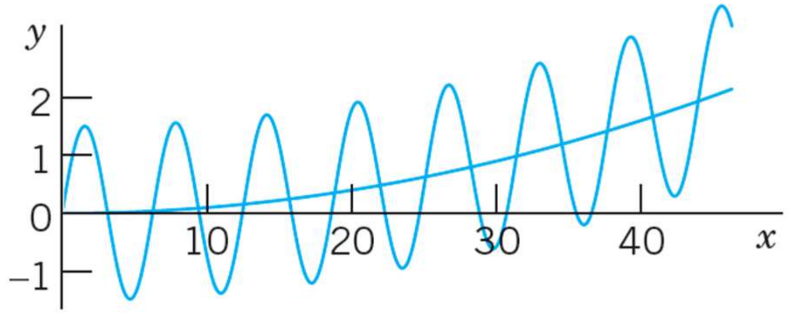

Fig. 50. Solution in Example 1

Example: Application of the Modification Rule (b)

- Solve the initial value problem

(6) $$y'' + 3y' + 2.25y = -10e^{-1.5x}, \; y(0) = 1, \; y'(0) = 0.$$

- ***Step 1. General solution of the homogeneous ODE.*** The characteristic equation of the homogeneous ODE is $\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$. Hence the homogeneous ODE has the general solution

$$y_h = (c_1 + c_2 x)e^{-1.5x}.$$

Example: Application of the Modification Rule (b)

- ***Step 2. Solution y_p of the nonhomogeneous ODE.*** The function $e^{-1.5x}$ on the right would normally require the choice $Ce^{-1.5x}$. But we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a *double root* of the characteristic equation. Hence, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose

$$y_p = Cx^2 e^{-1.5x}.$$

Then $y_p' = C(2x - 1.5x^2)\, e^{-1.5x}$, $y_p'' = C(2 - 3x - 3x + 2.25x^2)\, e^{-1.5x}$.

Example: Application of the Modification Rule (b)

- We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

  $C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10.$

- Comparing the coefficients of x^2, x, x^0 gives

  $0 = 0$, $0 = 0$, $2C = -10$, hence $C = -5$.

  This gives the solution $y_p = -5x^2e^{-1.5x}$. Hence the given ODE has the general solution

$$y = y_h + y_p = (c_1 + c_2x)e^{-1.5x} - 5x^2e^{-1.5x}.$$

Example: Application of the Modification Rule (b)

- ***Step 3. Solution of the initial value problem.*** Setting $x = 0$ in y and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of y gives

  $y' = (c_2 - 1.5c_1 - 1.5c_2x)e^{-1.5x} - 10x\,e^{-1.5x} + 7.5x^2e^{-1.5x}$.
- From this and the second initial condition we have

  $$y'(0) = c_2 - 1.5c_1 = 0. \text{ Hence } c_2 = 1.5c_1 = 1.5.$$

  This gives the answer (Fig. 51)

  $y = (1 + 1.5x)\,e^{-1.5x} - 5x^2\,e^{-1.5x} = (1 + 1.5x - 5x^2)\,e^{-1.5x}$.

  The curve begins with a horizontal tangent, crosses the x-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as x increases.

Example: Application of the Modification Rule (b)

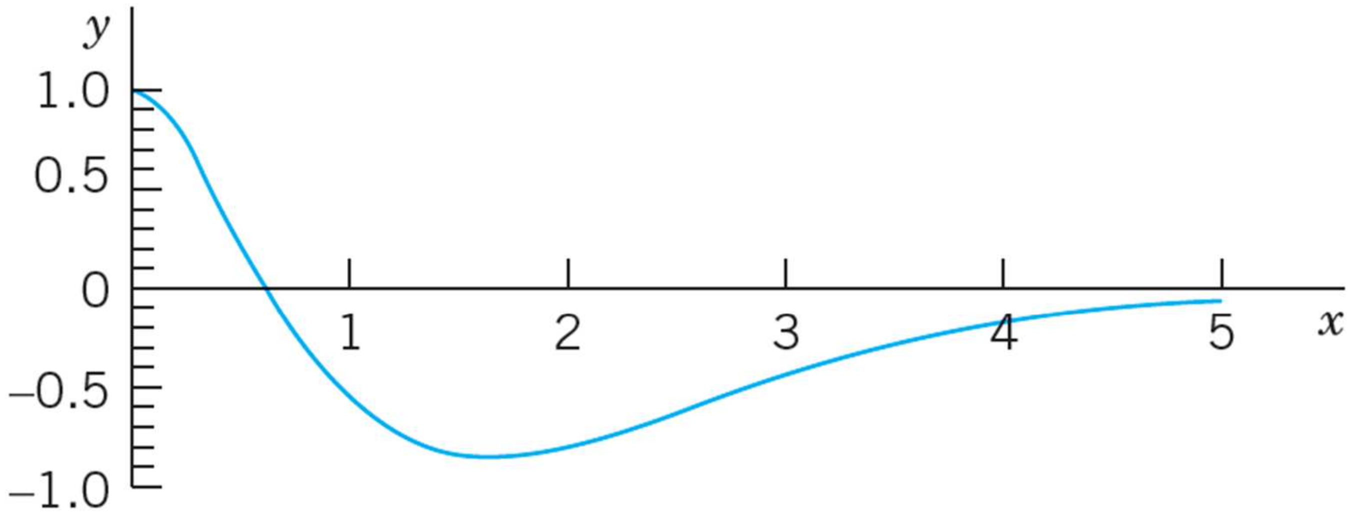

Fig. 51. Solution in Example 2

Example: Application of the Sum Rule (c)

- Solve the initial value problem

(7) $y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x, y(0) = 2.78, y'(0) = -0.43$

- Step 1. General solution of the homogeneous ODE. The characteristic equation of the homogeneous ODEs is $\lambda^2 + 2\lambda + 0.75 = (\lambda + \frac{1}{2})(\lambda + \frac{3}{2}) = 0$, which gives the general solution $y_h = c_1 e^{-x/2} + c_2 e^{-3x/2}$
- Step 2. Particular solution of the nonhomogeneous ODE. We write $y_p = y_{p1} + y_{p2}$ and, following Table 2.1, (C) and (B),

$$y_{p1} = K\cos x + M\sin x \qquad y_{p2} = K_1 x + K_0$$

Example: Application of the Sum Rule (c)

- Differentiation gives $y'_{p1} = -K \sin x + M \cos x$, $y''_{p1} = -K \cos x - M \sin x$ and $y'_{p2} = K_1, y''_{p2} = 0$. Substitution of y_{p1} into the ODE in (7) gives, by comparing the cosine and sine terms,

$$-K + 2M + 0.75K = 2 \qquad -M - 2K + 0.75M = -0.25$$

hence *K*=0 and *M*=1. Substituting y_{p2} into the ODE in (7) and comparing the *x*- and x^0-terms gives

$$0.75K_1 = 0.09 \qquad K_1 = 0.12$$
$$2K_1 + 0.75K_0 = 0 \qquad K_0 = -0.32$$

- Hence a general solution is

$$y = c_1 e^{-x/2} + c_2 e^{-3x/2} + \sin x + 0.12x - 0.32$$

$$y = c_1 e^{-x/2} + c_2 e^{-3x/2} + \sin x + 0.12x - 0.32$$

Example: Application of the Sum Rule (c)

- Step 3. Solution of the initial value problem. From *y*, *y*' and the initial conditions we obtain

$$y(0) = c_1 + c_2 - 0.32 = 2.78$$

$$y'(0) = -\tfrac{1}{2}c_1 - \tfrac{3}{2}c_2 + 1 + 0.12 = -0.4$$

Hence c_1=3.1, c_2=0. This gives the solution of the IVP

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32$$

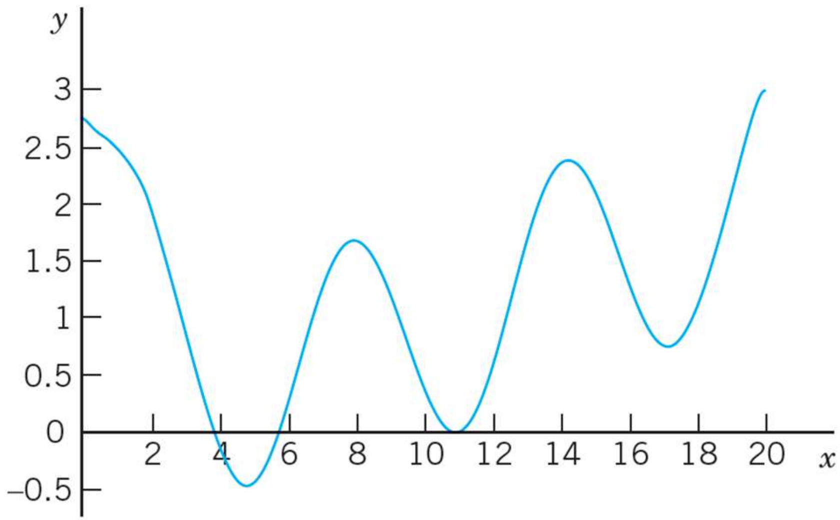

Fig. 52. Solution in Example 3

Stability

- If (and only if) all the roots of the characteristic equation of the homogeneous ODE $y'' + ay' + b = 0$ in (4) are negative, or have a negative real part, then a general solution y_h of this ODE goes to 0 as $x \to \infty$ so that the "**transient solution**" $y = y_h + y_p$ of (4) approaches the "**steady-state solution**" y_p. In this case the nonhomogeneous ODE and the physical or other system modeled by the ODE are called **stable**; otherwise they are called **unstable**. For instance, the ODE in Example 1 is unstable.

2.8 Modeling: Forced Oscillations. Resonance

Mass-Spring System

- In Sec. 2.4 we considered vertical motions of a mass–spring system (vibration of a mass *m* on an elastic spring, as in Figs. 33 and 53) and modeled it by the ***homogeneous*** linear ODE

(1) $$my'' + cy' + ky = 0.$$

- Here $y(t)$ as a function of time t is the displacement of the body of mass *m* from rest.

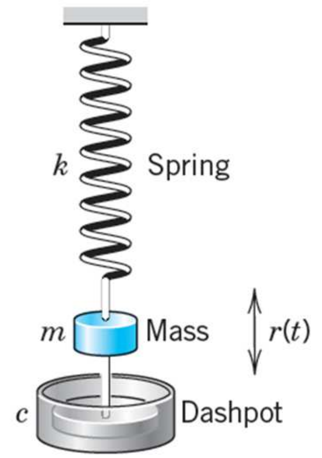

Fig. 53. Mass on a spring

Mass-Spring System

- The mass–spring system of Sec. 2.4 exhibited only free motion. This means no external forces (outside forces) but only internal forces controlled the motion.
- The internal forces are forces within the system. They are the force of inertia *my*", the damping force *cy*' (if *c* > 0), and the spring force *ky*, a restoring force.

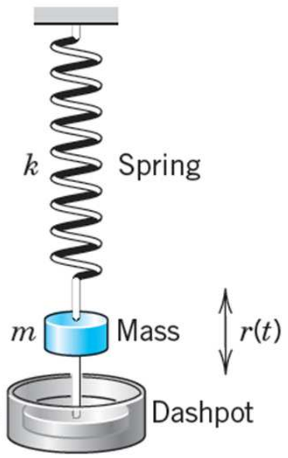

Fig. 53. Mass on a spring

Mass-Spring System

- We now extend our model by including an additional force, that is, the external force $r(t)$, on the right. Then we have

$$(2^*) \qquad my'' + cy' + ky = r(t).$$

- Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a **forced motion** with **forcing function** $r(t)$, which is also known as **input** or **driving force**, and the solution $y(t)$ to be obtained is called the **output** or the **response** *of the system to the driving force.*

Mass-Spring System

- Of special interest are periodic external forces, and we shall consider a driving force of the form

$$r(t) = F_0 \cos \omega t \qquad\qquad (F_0 > 0, \omega > 0).$$

- Then we have the nonhomogeneous ODE

(2) $$my'' + cy' + ky = F_0 \cos \omega t.$$

- Its solution will reveal facts that are fundamental in engineering mathematics and allow us to model resonance.

Solving the Nonhomogeneous ODE

- From Sec. 2.7 we know that a general solution of (2) is the sum of a general solution y_h of the homogeneous ODE (1) plus any solution y_p of (2). To find y_p, we use the method of undetermined coefficients (Sec. 2.7), starting from

(3) $$y_p(t) = a \cos \omega t + b \sin \omega t.$$

- By differentiating this function (chain rule) we obtain

$$y_p' = -\omega a \sin \omega t + \omega b \cos \omega t$$
$$y_p'' = -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t$$

Solving the Nonhomogeneous ODE

- Substituting y_p, y_p', y_p'' into (2) and collecting the cosine and the sine terms, we get

$$[(k - m\omega^2)a + \omega cb]\cos\omega t + [-\omega ca + (k - m\omega^2)b]\sin\omega t = F_0\cos\omega t$$

The cosine term on both sides must be equal, and the coefficient of the sine term on the left must be zero since there is no sine term on the right. This gives the two equations

$$\begin{aligned}(k - m\omega^2)a + \omega cb &= F_0 \\ -\omega ca + (k - m\omega^2)b &= 0\end{aligned} \qquad (4)$$

for determining the unknown coefficients *a* and *b*.

k: spring constant

$y_p(t) = a\cos\omega t + b\sin\omega t$

Solving the Nonhomogeneous ODE

- Solve by elimination.

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2} \qquad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}$$

- If we set $\sqrt{k/m} = \omega_0$ as in Sec. 2.4, then $k = m\omega_0^2$ and we obtain

(5) $$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$$

- We thus obtain the general solution of the nonhomogeneous ODE in the form

(6) $$y(t) = y_h(t) + y_p(t)$$

where y_h is a general solution of the homogeneous ODE (1) and y_p is given by (3) with coefficients (5)

Case 1. Undamped Forced Oscillations. Resonance

$$y_p(t) = a\cos\omega t + b\sin\omega t$$

- If the damping of the physical system is so small that its effect can be neglected over the time interval considered, we can set $c = 0$. Then (5) reduces to $a = F_0/[m(\omega_0^2 - \omega^2)]$ and $b = 0$. Hence (3) becomes (use $\omega_0^2 = k/m$)

(7) $$y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t = \frac{F_0}{k[1 - (\omega/\omega_0)^2]}\cos\omega t$$

- Here we must assume that $\omega^2 \neq \omega_0^2$; physically, the frequency $\omega/(2\pi)$ [cycles/sec] of the driving force is different from the *natural frequency* $\omega_0/(2\pi)$ of the system, which is the frequency of the free undamped motion [see (4) in Sec. 2.4].

Case 1. Undamped Forced Oscillations. Resonance

- From (7) and from (4*) in Sec. 2.4 we have the general solution of the "undamped system"

(8)
$$y(t) = C\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t$$

- *We see that this output is a **superposition of two harmonic oscillations** of the frequencies just mentioned.*

$$\boxed{y(t) = y_h(t) + y_p(t)}$$

Case 1. Undamped Forced Oscillations. Resonance

$$(7) \quad y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t = \frac{F_0}{k[1 - (\omega/\omega_0)^2]} \cos \omega t$$

- **Resonance.** We discuss (7). We see that the maximum amplitude of y_p is (put $\cos \omega t = 1$)

$$(9) \qquad a_0 = \frac{F_0}{k}\rho \qquad\qquad \rho = \frac{1}{1 - (\omega/\omega_0)^2}$$

- a_0 depends on ω and ω_0. If $\omega \to \omega_0$, then ρ and a_0 tend to infinity. This excitation of large oscillations by matching input and natural frequencies ($\omega = \omega_0$) is called **resonance**. ρ is called the **resonance factor** (Fig. 54), and from (9) we see that $\rho/k = a_0/F_0$ is the ratio of the amplitudes of the particular solution y_p and of the input $F_0 \cos \omega t$. We shall see later in this section that resonance is of basic importance in the study of vibrating systems.

Case 1. Undamped Forced Oscillations. Resonance

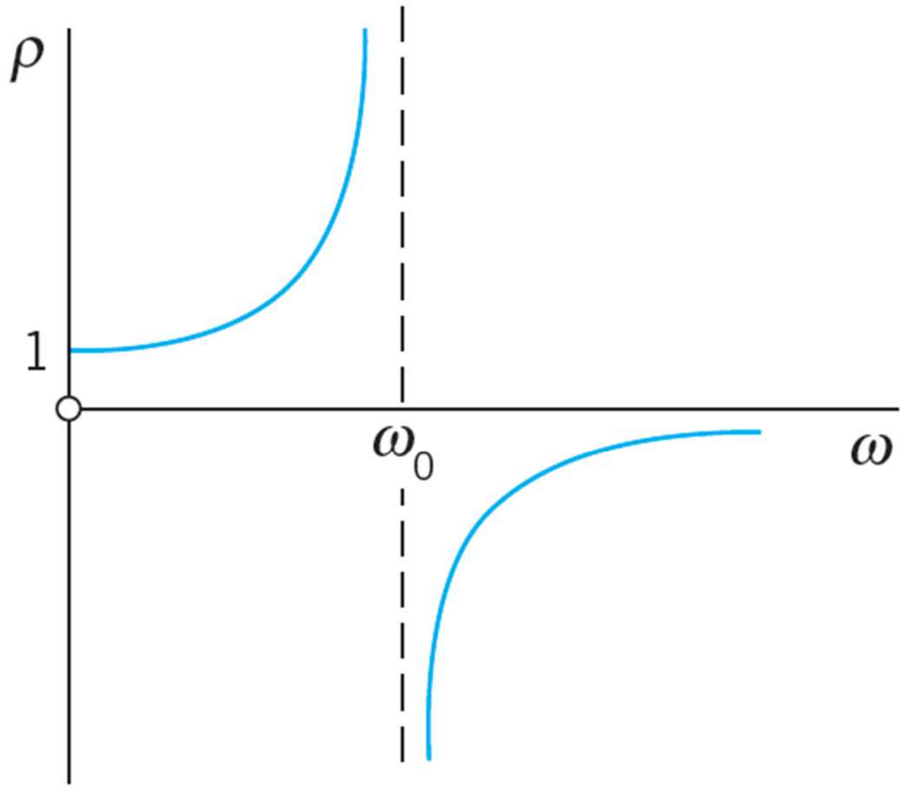

Fig. 54. Resonance factor $\rho(\omega)$

Case 1. Undamped Forced Oscillations. Resonance

- In the case of resonance the nonhomogeneous ODE (2) becomes

(10) $$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t$$

- Then (7) is no longer valid, and from the **Modification Rule** in Sec. 2.7, we conclude that a particular solution of (10) is of the form

$$y_p(t) = t(a \cos \omega_0 t + b \sin \omega_0 t).$$

The form of y_h is similar to that of y_p

$$my'' + cy' + ky = F_0 \cos \omega t$$

$\omega = \omega_0$
$c = 0$
$\Rightarrow$ $$y'' + \frac{k}{m} y = \frac{1}{m} F_0 \cos \omega_0 t$$

Case 1. Undamped Forced Oscillations. Resonance

- By substituting this into (10) we find $a = 0$ and $b = F_0/(2m\omega_0)$. Hence (Fig. 55)

(11)
$$y_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

- We see that, because of the factor t, the amplitude of the vibration becomes larger and larger. Practically speaking, systems with very little damping may undergo large vibrations that can destroy the system.

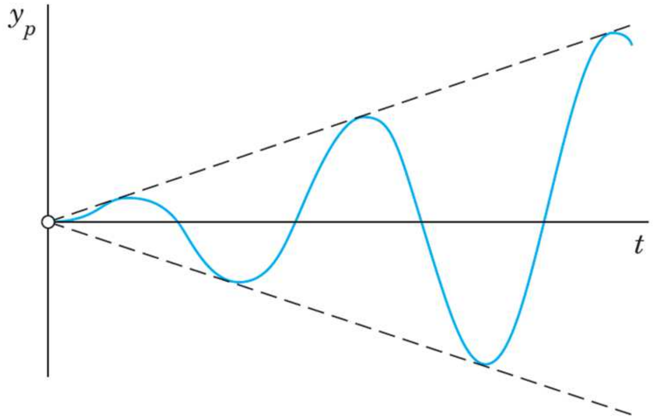

Fig. 55. Particular solution in the case of resonance

Case 1. Undamped Forced Oscillations. Resonance

- **Beats.** Another interesting and highly important type of oscillation is obtained if ω is close to ω_0. Take, for example, the particular solution [see (8)]

(12) $$y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t) \qquad \omega \neq \omega_0$$

- Using (12) in App. 3.1, we may write this as

$$y(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right)$$

Case 1. Undamped Forced Oscillations. Resonance

- Since is ω close to ω_0, the difference $\omega_0 - \omega$ is small. Hence the period of the last sine function is large, and we obtain an oscillation of the type shown in Fig. 56, the dashed curve resulting from the first sine factor. This is what musicians are listening to when they ***tune*** their instruments.

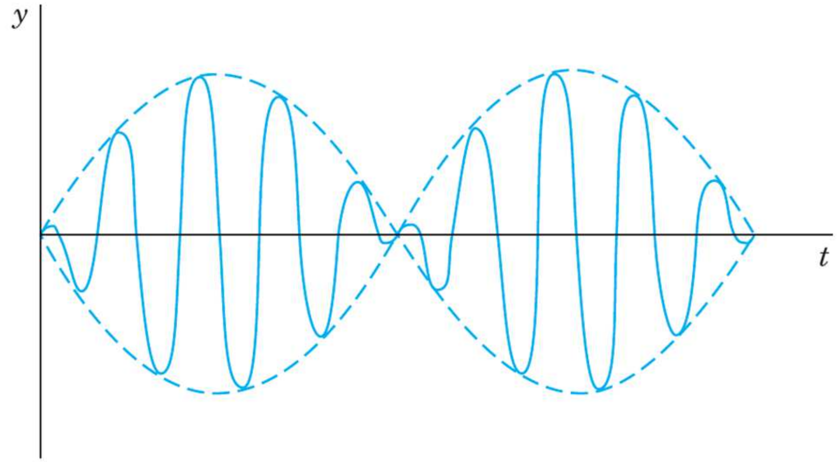

Fig. 56. Forced undamped oscillation when the difference of the input and natural frequencies is small (**"beats"**)

Case 2. Damped Forced Oscillations

- If the damping of the mass–spring system is not negligibly small, we have $c > 0$ and a damping term cy' in (1) and (2). Then the general solution y_h of the homogeneous ODE (1) approaches zero as t goes to infinity, as we know from Sec. 2.4.
- Practically, it is zero after a sufficiently long time. Hence the "**transient solution**" (6) of (2), given by $y = y_h + y_p$, approaches the "**steady-state solution**" y_p. This proves the following theorem.

Case 2. Damped Forced Oscillations

- **Theorem 1: Steady-State Solution**

- *After a sufficiently long time the output of a damped vibrating system under a purely sinusoidal driving force* [*see* (2)] *will practically be a harmonic oscillation whose frequency is that of the input.*

Case 2. Damped Forced Oscillations

- **Amplitude of the Steady-State Solution. Practical Resonance.**

- Whereas in the undamped case the amplitude of y_p approaches infinity as ω approaches ω_0, this will not happen in the damped case. In this case the amplitude will always be finite. But it may have a maximum for some ω depending on the damping constant c. This may be called **practical resonance**.

- It is of great importance because if c is not too large, then some input may excite oscillations large enough to damage or even destroy the system.

$$(3) \quad y_p(t) = a\cos\omega t + b\sin\omega t$$

Case 2. Damped Forced Oscillations

- To study the amplitude of y_p as a function of ω, we write (3) in the form

$$(13) \qquad y_p(t) = C^* \cos(\omega t - \eta).$$

- C^* is called the **amplitude** of y_p and η the **phase angle** or **phase lag** because it measures the lag of the output behind the input. According to (5), these quantities are

$$(14) \qquad C^*(\omega) = \sqrt{a^2 + b^2} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}}$$

$$\tan\eta(\omega) = \frac{b}{a} = \frac{\omega c}{m(\omega_0^2 - \omega)}$$

$$(5) \qquad a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2} \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$$

Case 2. Damped Forced Oscillations

- Let us see whether $C^*(\omega)$ has a maximum and, if so, find its location and then its size. We denote the radicand in the second root in *C** by *R*. Equating the derivative of *C** to zero, we obtain

$$\frac{dC^*}{d\omega} = F_0(-\tfrac{1}{2}R^{-3/2})[2m^2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega c^2]$$

The expression in the brackets [] is zero if

(15) $$c^2 = 2m^2(\omega_0^2 - \omega^2) \qquad (\omega_0^2 = k/m)$$

By reshuffling terms we have

$$2m^2\omega^2 = 2m^2\omega_0^2 - c^2 = 2mk - c^2$$

Case 2. Damped Forced Oscillations

- The right side of this equation becomes negative if $c^2 > 2mk$, so that then (15) has no real solution and *C** decreases monotone as ω increases, as the lowest curve in Fig. 57 shows. If *c* is smaller, $c^2 < 2mk$, then (15) has a real solution $\omega = \omega_{max}$, where

(15*) $$\omega_{max}^2 = \omega_0^2 - \frac{c^2}{2m^2}$$

From (15*) we see that this solution increases as *c* decreases and approaches ω_0 as *c* approaches zero. See also Fig. 57.

Case 2. Damped Forced Oscillations

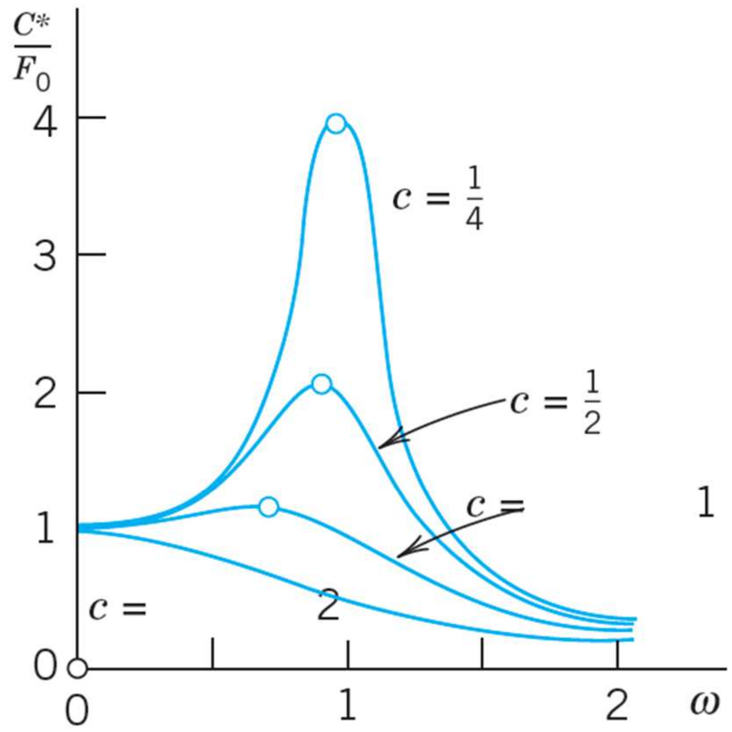

Fig. 57. Amplification C^*/F_0 as a function of ω for $m = 1, k = 1$, and various values of the damping constant c

Case 2. Damped Forced Oscillations

- The size of $C^*(\omega_{max})$ is obtained from (14), with $\omega^2 = \omega_{max}^2$ given by (15*). For this ω^2 we obtain in the second radicand in (14) from (15*)

  $$m^2(\omega_0^2 - \omega_{max}^2)^2 = \frac{c^4}{4m^2} \qquad \omega_{max}^2 c^2 = \left(\omega_0^2 - \frac{c^2}{2m^2}\right)c^2$$

  The sum of the right sides of these two formulas is

  $$(c^4 + 4m^2\omega_0^2 c^2 - 2c^4)/(4m^2) = c^2(4m^2\omega_0^2 - c^2)/(4m^2)$$

  Substitution into (14) gives

(16) $$C^*(\omega_{max}) = \frac{2mF_0}{c\sqrt{4m^2\omega_0^2 - c^2}}$$

Case 2. Damped Forced Oscillations

- We see that $C^*(\omega_{max})$ is always finite when $c > 0$. Furthermore, since the expression

$$c^2 4m^2\omega_0^2 - c^4 = c^2(4mk - c^2)$$

in the denominator of (16) decreases monotone to zero as c^2 ($<2mk$) goes to zero, the maximum amplitude (16) increases monotone to infinity, in agreement with our result in Case 1.

Case 2. Damped Forced Oscillations

- Figure 57 shows the **amplification** C^*/F_0 (ratio of the amplitudes of output and input) as a function of ω for $m = 1$, $k = 1$, hence $\omega_0 = 1$, and various values of the damping constant c.
- Figure 58 shows the phase angle (the lag of the output behind the input), which is less than $\pi/2$ when $\omega < \omega_0$, and greater than $\pi/2$ for $\omega > \omega_0$.

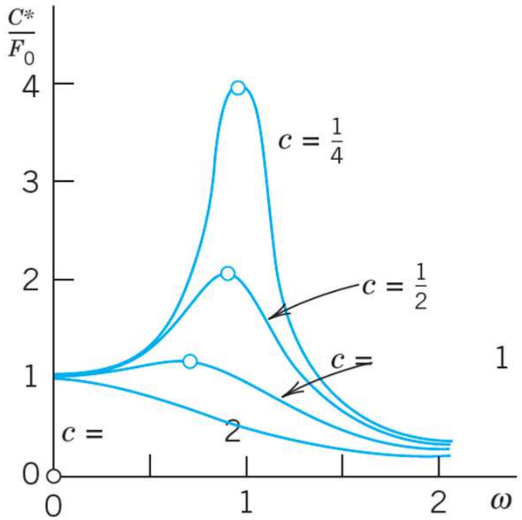

Fig. 57. Amplification C^*/F_0 as a function of ω for $m = 1, k = 1$, and various values of the damping constant c

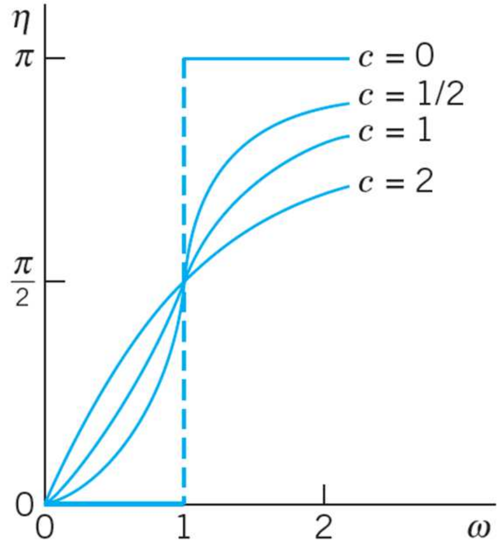

Fig. 58. Phase lag η as a function of ω for $m = 1, k = 1$, thus $\omega_0 = 1$, and various values of the damping constant c

2.9 Modeling: Electric Circuits

RLC Circuit

- Figure 61 shows an ***RLC*-circuit**, as it occurs as a basic building block of large electric networks in computers and elsewhere.
- An *RLC*-circuit is obtained from an *RL*-circuit by adding a capacitor. Recall Example 2 on the *RL*-circuit in Sec. 1.5: The model of the *RL*-circuit is $LI' + RI = E(t)$. It was obtained by **KVL** (Kirchhoff's Voltage Law)* by equating the voltage drops across the resistor and the inductor to the EMF (electromotive force).

- ***Kirchhoff's Current Law (KCL)**: *At any point of a circuit, the sum of the inflowing currents is equal to the sum of the outflowing currents.*

RLC Circuit

- Hence we obtain the model of the *RLC*-circuit simply by adding the voltage drop *Q/C* across the capacitor. Here, *C* F(farads) is the capacitance of the capacitor. *Q* coulombs is the charge on the capacitor, related to the current by

$$I(t) = \frac{dQ}{dt} \qquad Q(t) = \int I(t)dt$$

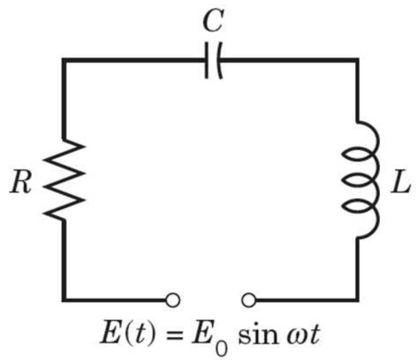

Fig. 61. *RLC*-circuit

Name	Symbol	Notation		Unit	Voltage Drop
Ohm's Resistor		R	Ohm's Resistance	ohms (Ω)	RI
Inductor		L	Inductance	henrys (H)	$L\frac{dI}{dt}$
Capacitor		C	Capacitance	farads (F)	Q/C

Fig. 62. Elements in an *RLC*-circuit

RLC Circuit

(1') $$LI' + RI + \frac{1}{C}\int I\,dt = E(t) = E_0 \sin \omega t$$

- This is an "integro-differential equation." To get rid of the integral, we differentiate (1') with respect to *t*, obtaining

(1) $$LI'' + RI' + \frac{1}{C}I = E'(t) = E_0 \omega \cos \omega t$$

- This shows that the current in an *RLC*-circuit is obtained as the solution of this nonhomogeneous second-order ODE (1) with constant coefficients. In connection with initial value problems, we shall occasionally use

(1") $$LQ'' + RQ'' + \frac{1}{C}Q = E(t)$$

obtained from (1') and $I = Q'$.

Solving the ODE (1)

- A general solution of (1) is the sum $I = I_h + I_p$, where I_h is a general solution of the homogeneous ODE corresponding to (1) and is a particular solution of (1).
- We first determine I_p by the method of undetermined coefficients, proceeding as in the previous section. We substitute

$$(2) \qquad \begin{aligned} I_p &= a \cos \omega t + b \sin \omega t \\ I_p' &= \omega(-a \sin \omega t + b \cos \omega t) \\ I_p'' &= \omega^2(-a \cos \omega t - b \sin \omega t) \end{aligned}$$

into (1).

Solving the ODE (1)

- Then we collect the cosine terms and equate them to $E_0\omega \cos \omega t$ on the right, and we equate the sine terms to zero because there is no sine term on the right,

  $L\omega^2(-a) + R\omega b + a/C = E_0\omega$ (Cosine terms)

  $L\omega^2(-b) + R\omega(-a) + b/C = 0$ (Sine terms).

- Before solving this system for a and b, we first introduce a combination of L and C, called the **reactance**

(3) $$S = \omega L - \frac{1}{\omega C}$$

Solving the ODE (1)

- Dividing the previous two equations by ω , ordering them, and substituting *S* gives

$$-Sa + Rb = E_0$$
$$-Ra - Sb = 0$$

We now eliminate *b* by multiplying the first equation by *S* and the second by *R*, and adding. Then we eliminate *a* by multiplying the first equation by *R* and the second by –*S*, and adding. This gives

$$-(S^2 + R^2)a = E_0 S$$
$$(R^2 + S^2)b = E_0 R$$

Solving the ODE (1)

- We can solve for *a* and *b*,

$$(4) \qquad a = \frac{-E_0 S}{R^2 + S^2} \qquad\qquad b = \frac{E_0 R}{R^2 + S^2}$$

- Equation (2) with coefficients *a* and *b* given by (4) is the desired particular solution I_p of the nonhomogeneous ODE (1) governing the current *I* in an *RLC*-circuit with sinusoidal electromotive force.

Solving the ODE (1)

- Using (4), we can write I_p in terms of "physically visible" quantities, namely, amplitude I_0 and phase lag θ of the current behind the EMF, that is,

(5) $$I_p(t) = I_0 \sin(\omega t - \theta)$$

where [see (14) in App. A3.1]

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}} \qquad \tan\theta = -\frac{a}{b} = \frac{S}{R}$$

- The quantity $\sqrt{R^2 + S^2}$ is called the **impedance**. Our formula shows that the impedance equals the ratio E_0/I_0. This is somewhat analogous to $E/I = R$ (Ohm's law), and because of this analogy, the impedance is also known as the **apparent resistance.**

Solving the ODE (1)

- A general solution of the homogeneous equation corresponding to (1) is

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

where λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

Solving the ODE (1)

- We can write these roots in the form $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = -\alpha - \beta$, where

$$\alpha = \frac{R}{2L} \qquad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$$

- Now in an actual circuit, R is never zero (hence $R > 0$). From this it follows that I_h approaches zero, theoretically $t \to \infty$, as but practically after a relatively short time. Hence the transient current $I = I_h + I_p$ tends to the steady-state current I_p, and after some time the output will practically be a harmonic oscillation, which is given by (5) and whose frequency is that of the input (of the electromotive force).

Example: RLC-Circuit

- Find the current $I(t)$ in an RLC-circuit with $R = 11\Omega$ (ohms), L = 0.1 H(henry), C=10^{-2} F (farad), which is connected to a source of EMF $E(t) = 110 \sin(60 \cdot 2\pi t) = 110 \sin 377t$ (hence 60 Hz = 60 cycles/sec). Assume that current and capacitor charge are 0 when t = 0.

Example: RLC-Circuit

- Step 1. General solution of the homogeneous ODE. Substituting *R*, *L*, *C* and the derivative *E'*(*t*) into (1), we obtain

$$0.1I'' + 11I' + 100I = 100 \cdot 377 \cos 377t$$

Hence the homogeneous ODE is $0.1I'' + 11I' + 100I = 0$
Its characteristic equation is

$$0.1\lambda^2 + 11\lambda + 100 = 0$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is

$$I_h(t) = c_1 e^{-10t} + c_2 e^{-100t}$$

Example: RLC-Circuit

- Step 2. Particular solution I_p of (1). We calculate the reactance *S*=37.7-0.3=37.4 and the steady-state current

$$I_p(t) = a\cos 377t + b\sin 377t$$

with coefficients obtained from (4) (and rounded)

$$a = \frac{-100 \cdot 37.4}{11^2 + 37.4^2} = -2.71 \qquad b = \frac{110 \cdot 11}{11^2 + 37.4^2} = 0.796$$

Hence in our present case, a general solution of the nonhomogeneous ODE(1) is

(6) $$I(t) = c_1 e^{-10t} + c_2 e^{-100t} - 2.71\cos 377t + 0.796\sin 377t$$

Example: RLC-Circuit

- Step 3. Particular solution satisfying the initial conditions. How to use $Q(0)=0$? We finally determine c_1 and c_2 from the initial conditions $I(0) = 0$ and $Q(0) = 0$. From the first condition and (6) we have

(7) $$I(0) = c_1 + c_2 - 2.71 = 0 \qquad c_2 = 2.71 - c_1$$

We turn to $Q(0) = 0$. The integral in (1') equals $\int I dt = Q(t)$ Hence for $t = 0$, Eq. (1') becomes

$$LI'(0) = R \cdot 0 = 0 \qquad I'(0) = 0$$

Differentiating (6) and setting $t = 0$, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.796 \cdot 377 = 0$$

by (7), $10c_1 = 100(2.71 - c_1) - 300.1$

Example: RLC-Circuit

- The solution of this and (7) is c_1=-0.323, c_2=3.033. Hence the answer is

$$I(t) = -0.323e^{-10t} + 3.033e^{-100t} - 2.71\cos 377t + 0.796\sin 377t$$

- Figure 63 shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near t=0 because the exponential terms go to zero very rapidly. Thus after a very short time the current will practically execute harmonic oscillations of the input frequency 60 Hz. Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.824\sin(377t - 1.29)$$

Example: RLC-Circuit

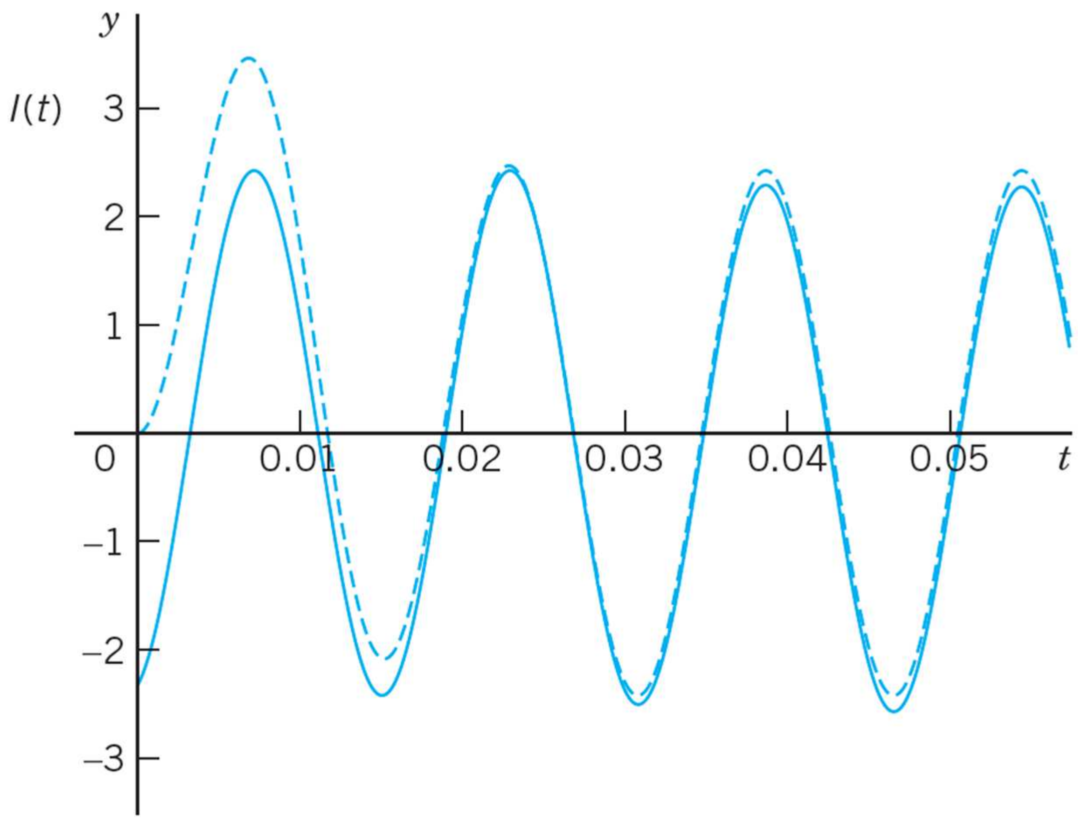

Fig. 63. Transient (upper curve) and steady-state currents in Example 1

Analogy of Electrical and Mechanical Quantities

- ***Entirely different physical or other systems may have the same mathematical model.*** For instance, we have seen this from the various applications of the ODE *y*' = *ky* in Chap. 1. Another impressive demonstration of this ***unifying power of mathematics*** is given by the ODE (1) for an electric *RLC*-circuit and the ODE (2) in the last section for a mass–spring system. Both equations
  $LI'' + RI' + \frac{1}{C}I = E_0\omega \cos \omega t$ and $my'' + cy' + ky = F_0 \cos \omega t$
  are of the same form.

Analogy of Electrical and Mechanical Quantities

- Table 2.2 shows the analogy between the various quantities involved. The inductance *L* corresponds to the mass *m* and, indeed, an inductor opposes a change in current, having an "inertia effect" similar to that of a mass. The resistance *R* corresponds to the damping constant *c*, and a resistor causes loss of energy, just as a damping dashpot does. And so on.
- This analogy is ***strictly quantitative*** in the sense that to a given mechanical system we can construct an electric circuit whose current will give the exact values of the displacement in the mechanical system when suitable scale factors are introduced.

Analogy of Electrical and Mechanical Quantities

- The ***practical importance*** of this analogy is almost obvious. The analogy may be used for constructing an "electrical model" of a given mechanical model, resulting in substantial savings of time and money because electric circuits are easy to assemble, and electric quantities can be measured much more quickly and accurately than mechanical ones.

Analogy of Electrical and Mechanical Quantities

Table 2.2 Analogy of Electrical and Mechanical Quantities

Electrical System	Mechanical System
Inductance L	Mass m
Resistance R	Damping constant c
Reciprocal $1/C$ of capacitance	Spring modulus k
Derivative $E_0\omega \cos \omega t$ of electromotive force	Driving force $F_0 \cos \omega t$
Current $I(t)$	Displacement $y(t)$

2.10 Solution by Variation of Parameters

Solution by Variation of Parameters

- We continue our discussion of nonhomogeneous linear ODEs, that is

(1) $$y'' + p(x)y' + q(x)y = r(x).$$

- To obtain y_p when $r(x)$ is not too complicated, we can often use the *method of undetermined coefficients*. However, since this method is restricted to functions $r(x)$ whose derivatives are of a form similar to $r(x)$ itself (powers, exponential functions, etc.), it is desirable to have a method valid for more general ODEs (1), which we shall now develop.
- It is called the **method of variation of parameters** and is credited to Lagrange (Sec. 2.1). Here p, q, r in (1) may be variable (given functions of x), but we assume that they are continuous on some open interval I.

Solution by Variation of Parameters

- Lagrange's method gives a particular solution y_p of (1) on I in the form

(2) $$y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

where y_1, y_2 form a basis of solutions of the corresponding homogeneous ODE

(3) $$y'' + p(x)y' + q(x)y = 0$$

on I, and W is the Wronskian of y_1, y_2.

(4) $$W = y_1 y_2' - y_2 y_1'$$ (see Sec. 2.6).

Solution by Variation of Parameters

- CAUTION! The solution formula (2) is obtained under the assumption that the ODE is written in standard form, with *y*'' as the first term as shown in (1). If it starts with *f*(*x*)*y*'', divide first by *f*(*x*).
- The integration in (2) may often cause difficulties, and so may the determination of y_1, y_2 if (1) has variable coefficients. If you have a choice, use the previous method. It is simpler.

$$(2) \qquad y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

Example: Method of Variation of Parameters

(2) $$y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

- Solve the nonhomogeneous ODE

$$y'' + y = \sec x = \frac{1}{\cos x}$$

- ***Solution.*** A basis of solutions of the homogeneous ODE on any interval is $y_1 = \cos x$, $y_2 = \sin x$. This gives the Wronskian

$$W(y_1, y_2) = \cos x \cos x - \sin x\,(-\sin x) = 1.$$

- From (2), choosing zero constants of integration, we get the particular solution of the given ODE

$$y_p = -\cos x \int \sin x \sec x dx + \sin x \int \cos x \sec x dx$$
$$= \cos x \ln|\cos x| + x \sin x$$

$$-\cos x \int \sin x \sec x dx$$
$$= \cos x \int \frac{1}{u} du$$
$$= \cos x \ln|\cos x|$$
$$u = \cos x, du = -\sin x dx$$

Example: Method of Variation of Parameters

- Figure 70 shows y_p and its first term, which is small, so that $x \sin x$ essentially determines the shape of the curve of y_p. From y_p and the general solution $y_h = c_1y_1 + c_2y_2$ of the homogeneous ODE we obtain the a*nswer*

$$y = y_h + y_p = (c_1 + \ln|\cos x|) \cos x + (c_2 + x) \sin x.$$

- Had we included integration constants $-c_1$, c_2 in (2), then (2) would have given the additional $c_1\cos x + c_2\sin x = c_1y_1 + c_2y_2$, that is, a general solution of the given ODE directly from (2). This will always be the case.

Example: Method of Variation of Parameters

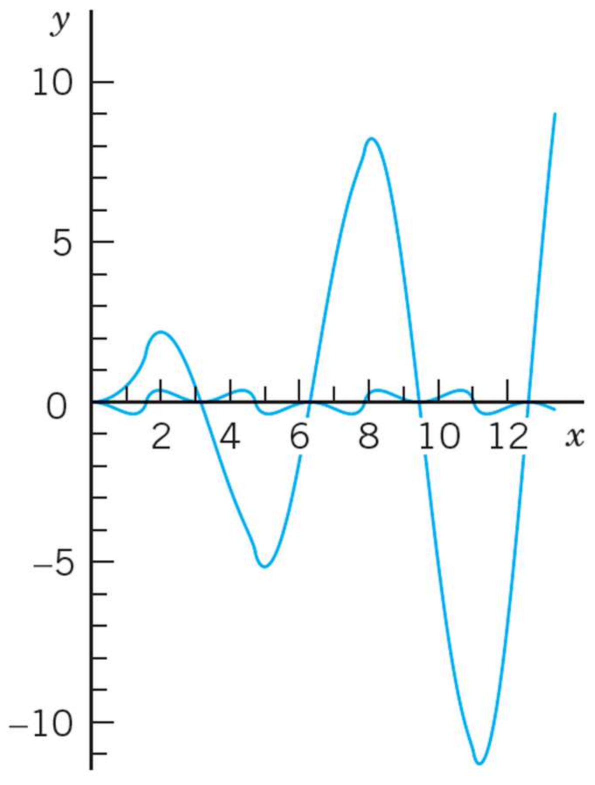

Fig. 70. Particular solution y_p and its first term in Example 1

$$(2) \quad y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

Derivation of (2)

- The idea is to start from a general solution
$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$
of the homogeneous ODE (3) on an open interval *I* and to replace the constants c_1 and c_2 by functions $u(x)$ and $v(x)$; this suggests the name of the method. We shall determine *u* and *v* so that the resulting function

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) \quad (5)$$

is a particular solution of the nonhomogeneous ODE (1). Note that y_h exists by Theorem 3 in Sec. 2.6 because of continuity of *p* and *q* on *I*.

(5) $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$

Derivation of (2)

- We determine u and v by substituting (5) and its derivatives into (1). Differentiating (5), we obtain

  $$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'$$

  Now y_p must satisfy (1). This is one condition for two functions u and v. It seems plausible that we may impose a second condition. Indeed, our calculation will show that we can determine u and v such that y_p satisfies (1) and u and v satisfy as a second condition the equation

(6) $$u'y_1 + v'y_2 = 0$$

Derivation of (2)

- This reduces the first derivative y_p' to the simpler form

(7) $$y_p' = uy_1' + vy_2'$$

Differentiating (7), we obtain

(8) $$y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2''$$

We now substitute y_p and its derivatives according to (5), (7), (8) into (1). Collecting terms in u and terms in v, we obtain

$$u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + u'y_1' + v'y_2' = r$$

Since y_1 and y_2 are solutions of the homogeneous ODE (3), this reduces to

(9a) $$u'y_1' + v'y_2' = r$$

Derivation of (2)

(9a) $u'y_1' + v'y_2' = r$

- Equation (6) is

(9b) $$u'y_1 + v'y_2 = 0$$

This is a linear system of two algebraic equations for the unknown functions *u*' and *v*'. We can solve it by elimination as follows. To eliminate *v*', we multiply (9a) by $-y_2$ and (9b) by y_2' and add, obtaining

$$u'(y_1y_2' - y_2y_1') = -y_2r \qquad u'W = -y_2r$$

Here, *W* is the Wronskian (4) of y_1, y_2. To eliminate *u*' we multiply (9a) by y_1, and (9b) by $-y_1$' and add, obtaining

$$v'(y_1y_2' - y_2y_1') = -y_1r \qquad v'W = y_1r$$

Derivation of (2)

- Since y_1, y_2 form a basis, we have $W \neq 0$ (by Theorem 2 in Sec. 2.6) and can divide by *W*,

(10) $$u' = -\frac{y_2 r}{W} \qquad v' = \frac{y_1 r}{W}$$

By integration,

$$u = -\int \frac{y_2 r}{W} dx \qquad v = \int \frac{y_1 r}{W} dx$$

These integrals exist because *r*(x) is continuous. Inserting them into (5) gives (2) and completes the derivation.

(5) $$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

(2) $$y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

Summary of Chapter 2

Summary

- Second-order linear ODEs are particularly important in applications, for instance, in mechanics (Secs. 2.4, 2.8) and electrical engineering (Sec. 2.9). A second-order ODE is called **linear** if it can be written

(1) $$y'' + p(x)y' + q(x)y = r(x)$$ (Sec. 2.1).

- (If the first term is, say, $f(x)y''$, divide by $f(x)$ to get the "**standard form**" (1) with y'' as the first term.) Equation (1) is called **homogeneous** if $r(x)$ is zero for all x considered, usually in some open interval; this is written $r \equiv 0$. Then

(2) $$y'' + p(x)y' + q(x)y = 0$$

- Equation (1) is called **nonhomogeneous** if $r(x) \equiv 0$ (meaning $r(x)$ is not zero for some x considered).

Summary

- For the homogeneous ODE (2) we have the important **superposition principle** (Sec. 2.1) that a linear combination $y = ky_1 + ly_2$ of two solutions y_1, y_2 is again a solution.
- Two *linearly independent* solutions y_1, y_2 of (2) on an open interval I form a **basis** (or **fundamental system**) of solutions on I, and $y = c_1y_1 + c_2y_2$ with arbitrary constants c_1, c_2 a **general solution** of (2) on I. From it we obtain a **particular solution** if we specify numeric values (numbers) for c_1 and c_2, usually by prescribing two **initial conditions**

(3) $y(x_0) = K_0, \;\; y'(x_0) = K_1$

x_0, K_0, K_1 given numbers; Sec. 2.1).

(2) and (3) together form an **initial value problem**. Similarly for (1) and (3).

Summary

- For a nonhomogeneous ODE (1) a **general solution** is of the form

(4) $y = y_h + y_p$ (Sec. 2.7).

- Here y_h is a general solution of (2) and y_p is a particular solution of (1). Such a y_p can be determined by a general method (***variation of parameters***, Sec. 2.10) or in many practical cases by the ***method of undetermined coefficients***. The latter applies when (1) has constant coefficients p and q, and $r(x)$ is a power of x, sine, cosine, etc. (Sec. 2.7). Then we write (1) as

(5) $y'' + ay' + by = r(x)$ (Sec. 2.7).

- The corresponding homogeneous ODE $y' + ay' + by = 0$ has solutions $y = e^{\lambda x}$ where λ is a root of

(6) $\lambda^2 + a\lambda + b = 0.$

Summary

- Hence there are three cases (Sec. 2.2):

Case	Type of Roots	General Solution
I	Distinct real λ_1, λ_2	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Double $-\frac{1}{2}a$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex $-\frac{1}{2}a \pm i\omega^*$	$y = e^{-ax/2}(A \cos \omega^* x + B \sin \omega^* x)$

- Here ω^* is used since ω is needed in driving forces.
- Important applications of (5) in mechanical and electrical engineering in connection with ***vibrations*** and ***resonance*** are discussed in Secs. 2.4, 2.7, and 2.8.

Summary

- Another large class of ODEs solvable "algebraically" consists of the **Euler–Cauchy equations**

(7) $x^2y'' + axy' + by = 0$ (Sec. 2.5).

- These have solutions of the form $y = x^m$, where m is a solution of the auxiliary equation

(8) $m^2 + (a - 1)m + b = 0.$

- ***Existence and uniqueness*** of solutions of (1) and (2) is discussed in Secs. 2.6 and 2.7, and ***reduction of order*** in Sec. 2.1.