Chapter 11 Fourier Analysis

Advanced Engineering Mathematics

Wei-Ta Chu
National Chung Cheng University
wtchu@cs.ccu.edu.tw
11.1 Fourier Series
Fourier Series

- Fourier series are infinite series that represent periodic functions in terms of cosines and sines. As such, Fourier series are of greatest importance to the engineer and applied mathematician.

- A function $f(x)$ is called a **periodic function** if $f(x)$ is defined for all real $x$, except possibly at some points, and if there is some positive number $p$, called a **period** of $f(x)$, such that

\[(1) \quad f(x + p) = f(x)\]
Fourier Series

- The graph of a periodic function has the characteristic that it can be obtained by periodic repetition of its graph in any interval of length $p$. (Fig. 258)
- The smallest positive period is often called the fundamental period.
Fourier Series

- If $f(x)$ has period $p$, it also has the period $2p$ because (1) implies $f(x + 2p) = f([x + p] + p) = f(x + p) = f(x)$; thus for any integer $n=1, 2, 3, \ldots$

(2) \[ f(x + np) = f(x) \]

Furthermore if $f(x)$ and $g(x)$ have period $p$, then $af(x) + bg(x)$ with any constants $a$ and $b$ also has the period $p$. 
Fourier Series

- Our problem in the first few sections of this chapter will be the representation of various functions $f(x)$ of period $2\pi$ in terms of the simple functions

$\begin{align*}
1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots
\end{align*}$

- All these functions have the period $2\pi$. They form the so-called trigonometric system. Figure 259 shows the first few of them.
Fourier Series

The series to be obtained will be a trigonometric series, that is, a series of the form

\[ a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots \]

\[ = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \]

\(a_0, a_1, b_1, a_2, b_2, \ldots\) are constants, called the coefficients of the series. We see that each term has the period \(2\pi\). Hence if the coefficients are such that the series converges, its sum will be a function of period \(2\pi\).
Fourier Series

Now suppose that $f(x)$ is a given function of period $2\pi$ and is such that it can be represented by a series (4), that is, (4) converges and, moreover, has the sum $f(x)$. Then, using the equality sign, we write

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and call (5) the **Fourier series** of $f(x)$. We shall prove that in this case the coefficients of (5) are the so-called **Fourier coefficients** of $f(x)$, given by the **Euler formulas**

\[
\begin{align*}
(6.0) \quad a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \\
(6.a) \quad a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\
(6.b) \quad b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx
\end{align*}
\]
Example 1

- Find the Fourier coefficients of the period function $f(x)$ in Fig. 260. The formula is

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases}$$

$$f(x + 2\pi) = f(x)$$

Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm \pi$

![Fig. 260. Given function $f(x)$ (Periodic rectangular wave)]
Example 1

- From (6.0) we obtain $a_0 = 0$. This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ and $\pi$ is zero. From (6.a) we obtain the coefficient $a_1, a_2, \ldots$ of the cosine terms. Since $f(x)$ is given by two expressions, the integrals from $-\pi$ to $\pi$ split into two integrals:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nxdx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} (-k) \cos nxdx + \int_{0}^{\pi} k \cos nxdx \right]$$

$$= \frac{1}{\pi} \left[ k \left( \frac{\sin nx}{n} \right)_{-\pi}^{0} - k \left( \frac{\sin nx}{n} \right)_{0}^{\pi} \right] = 0$$

because $\sin nx = 0$ at $-\pi, 0, \pi$ for all $n = 1, 2, \ldots$. 

\[(6.0) \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \]

\[(6.a) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \]

\[(6.b) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \]
Example 1

- We see that all these cosine coefficients are zero. That is, the Fourier series of (7) has no cosine terms, just sine terms, it is a Fourier sine series with coefficients \( b_1, b_2, \ldots \) obtained from (6b);

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} (-k) \sin nx \, dx + \int_{0}^{\pi} k \sin nx \, dx \right]
\]

\[
= \frac{1}{\pi} \left[ k \frac{\cos nx}{n} \bigg|_{-\pi}^{0} - k \frac{\cos nx}{n} \bigg|_{0}^{\pi} \right]
\]

Since \( \cos(-\alpha) = \cos \alpha, \cos 0 = 1 \), this yields

\[
b_n = \frac{k}{n\pi} \left[ \cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0 \right] = \frac{2k}{n\pi}(1 - \cos n\pi)
\]
Example 1

Now, \( \cos \pi = -1, \cos 2\pi = 1, \cos 3\pi = -1 \), etc.; in general,

\[
\cos n\pi = \begin{cases} 
-1 & \text{for odd } n \\
1 & \text{for even } n 
\end{cases}
\]

\[
1 - \cos n\pi = \begin{cases} 
2 & \text{for odd } n \\
0 & \text{for even } n 
\end{cases}
\]

Hence the Fourier coefficients \( b_n \) of our function are

\[
b_1 = \frac{4k}{\pi}, b_2 = 0, b_3 = \frac{4k}{3\pi}, b_4 = 0, b_5 = \frac{4k}{5\pi}, ...
\]

\[
b_n = \frac{k}{n\pi} [\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{n\pi} (1 - \cos n\pi)
\]
Example 1

- Since the $a_n$ are zero, the Fourier series of $f(x)$ is

\[
\frac{4k}{\pi} (\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots)
\]

\[(8)\]

The partial sums are

\[S_1 = \frac{4k}{\pi} \sin x \quad S_2 = \frac{4k}{\pi} \left( \sin x + \frac{1}{3} \sin 3x \right)\]

Their graphs in Fig. 261 seem to indicate that the series is convergent and has the sum $f(x)$, the given function. We notice that at $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the limits $-k$ and $k$ of our function, at these points. This is typical.
Example 1

- Furthermore, assuming that \( f(x) \) is the sum of the series and setting \( x = \pi/2 \), we have

\[
f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right)
\]

Thus

\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}
\]

This is a famous result obtained by Leibniz in 1673 from geometric considerations. It illustrates that the values of various series with constant terms can be obtained by evaluating Fourier series at specific points.
Example 1

Fig. 261. First three partial sums of the corresponding Fourier series
Theorem 1 Orthogonality of the Trigonometric System

- The trigonometric system (3) is orthogonal on the interval $-\pi \leq x \leq \pi$ (hence also on $0 \leq x \leq 2\pi$ or any other interval of length $2\pi$ because of periodicity); that is, the integral of the product of any two functions in (3) over that interval is 0, so that for any integers $n$ and $m$,

  (a) $\int_{-\pi}^{\pi} \cos nx \cos mxdx = 0 \quad (n \neq m)$

  (9) (b) $\int_{-\pi}^{\pi} \sin nx \sin mxdx = 0 \quad (n \neq m)$

  (c) $\int_{-\pi}^{\pi} \sin nx \cos mxdx = 0 \quad (n \neq m \text{ or } n = m)$

(3) $1, \cos x, \sin x, \cos 2x, \sin 2x, ..., \cos nx, \sin nx, ...$
Proof of Theorem 1

- Simply by transforming the integrands trigonometrically from products into sums

\[
\int_{-\pi}^{\pi} \cos nx \cos mxdx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n + m)x dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos(n - m)x dx
\]
\[
\int_{-\pi}^{\pi} \sin nx \sin mxdx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n - m)x dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos(n + m)x dx
\]

Since \( m \neq n \) (integer!), the integrals on the right are all 0. Similarly, for all integers

\[
\int_{-\pi}^{\pi} \sin nx \cos mxdx = \frac{1}{2} \int_{-\pi}^{\pi} \sin(n + m)x dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin(n - m)x dx = 0 + 0
\]

\[
\cos x \cos y = \frac{1}{2} \left[ \cos(x + y) + \cos(x - y) \right]
\]
\[
\sin x \sin y = \frac{1}{2} \left[ - \cos(x + y) + \cos(x - y) \right]
\]
\[
\sin x \cos y = \frac{1}{2} \left[ \sin(x + y) + \sin(x - y) \right]
\]
Application of Theorem 1 to the Fourier Series (5)

- We prove (6.0). Integrating on both sides of (5) from $-\pi$ to $\pi$, we get
  \[ \int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} \left[ a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \, dx \]

  We now assume that termwise integration is allowed. Then we obtain
  \[ \int_{-\pi}^{\pi} f(x) \, dx = a_0 \int_{-\pi}^{\pi} \, dx + \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \, dx \right) \]

  The first term on the right equals $2\pi a_0$. Integration shows that all the other integrals are 0. Hence division by $2\pi$ gives (6.0).

\[ (6.0) \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \]
Application of Theorem 1 to the Fourier Series (5)

(6.a) \[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \]

- We prove (6.a). Multiplying (5) on both sides by \( \cos mx \) with any fixed positive integer \( m \) and integrating from \(-\pi\) to \( \pi \), we have

\[
\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi} \left[ a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \cos mx \, dx
\]

We now integrate term by term. Then on the right we obtain an integral of \( a_0 \cos mx \), which is 0; an integral of \( a_n \cos nx \cos mx \), which is \( a_m \pi \) for \( n = m \) and 0 for \( n \neq m \) by (9a); and an integral of \( b_n \sin nx \cos mx \), which is 0 for all \( n \) and \( m \) by (9c). Hence the right side of (10) equals \( a_m \pi \). Division by \( \pi \) gives (6a) (with \( m \) instead of \( n \)).

\[
\int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{4} \sin 2x + c
\]
Application of Theorem 1 to the Fourier Series (5)

(6.b) \[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \]

- We finally prove (6.b). Multiplying (5) on both sides by $\sin mx$ with any fixed positive integer $m$ and integrating from $-\pi$ to $\pi$, we get

\[ \int_{-\pi}^{\pi} f(x) \sin mx \, dx = \int_{-\pi}^{\pi} \left[ a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \sin mx \, dx \]

Integrating term by term, we obtain on the right an integral of $a_0 \sin mx$, which is 0; an integral of $a_n \cos nx \sin mx$, which is 0 by (9c); and an integral $b_n \sin nx \sin mx$, which is $b_m \pi$ if $n = m$ and 0 if $n \neq m$, by (9b). This implies (6b) (with $n$ denoted by $m$). This completes the proof of the Euler formulas (6) for the Fourier coefficients.

\[ \int \sin^2 x \, dx = \frac{1}{2} x - \frac{1}{4} \sin 2x + c \]
Convergence and Sum of a Fourier Series

- The class of functions that can be represented by Fourier series is surprisingly large and general.

- Theorem 2: Let $f(x)$ be periodic with period $2\pi$ and piecewise continuous in the interval $-\pi \leq x \leq \pi$. Furthermore, let $f(x)$ have a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series (5) of $f(x)$ [with coefficients (6)] converges. Its sum is $f(x)$, except at points $x_0$ where $f(x)$ is discontinuous. There the sum of the series is the average of the left- and right-hand limits of $f(x)$ at $x_0$. 
Convergence and Sum of a Fourier Series

- The **left-hand limit** of \( f(x) \) at \( x_0 \) is defined as the limit of \( f(x) \) as \( x \) approaches \( x_0 \) from the left and is commonly denoted by \( f(x_0^-) \). Thus
  \[
  f(x_0^-) = \lim_{h \to 0^-} f(x_0 - h) \text{ as } h \to 0 \text{ through positive values.}
  \]

- The **right-hand limit** is denoted by \( f(x_0^+) \) and
  \[
  f(x_0^+) = \lim_{h \to 0^+} f(x_0 + h) \text{ as } h \to 0 \text{ through positive values.}
  \]

- The **left- and right-hand derivatives** of \( f(x) \) at \( x_0 \) are defined as the limits of
  \[
  \frac{f(x_0-h) - f(x_0)}{-h} \quad \text{and} \quad \frac{f(x_0+h) - f(x_0)}{h}
  \]
  respectively, as \( h \to 0 \) through positive values. Of course if \( f(x) \) is continuous at \( x_0 \), the last term in both numerators is simply \( f(x_0) \).
Example 2

- The rectangular wave in Example 1 has a jump at $x = 0$. Its left-hand limit there is $-k$ and its right-hand limit is $k$ (Fig. 261). Hence the average of these limits is 0. The Fourier series (8) of the wave does indeed converge to this value when $x=0$ because then all its terms are 0. Similarly for the other jumps. This is in agreement with Theorem 2.

Fig. 261. First three partial sums of the corresponding Fourier series
Summary

- A Fourier series of a given function $f(x)$ of period $2\pi$ is a series of the form (5) with coefficients given by the Euler formulas (6). Theorem 2 gives conditions that are sufficient for this series to converge and at each $x$ to have the value $f(x)$, except at discontinuities of $f(x)$, where the series equals the arithmetic mean of the left-hand and right-hand limits of $f(x)$ at that point.

\[
(5) \quad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

\[
(6.0) \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx
\]

\[
(6.a) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx
\]

\[
(6.b) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx
\]
11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions
Introduction

• This section concerns three topics:
  • Transition from period $2\pi$ to any period $2L$, for the function $f$, simply by a transformation of scale on the $x$-axis.
  • Simplifications. Only cosine terms if $f$ is even (“Fourier cosine series”). Only sine terms if $f$ is odd (“Fourier sine series”).
  • Expansion of $f$ given for $0 \leq x \leq L$ in two Fourier series, one having only cosine terms and the other only sine terms (“half-range expansions”).
Transition

- Periodic functions in applications may have any period. The notation $p=2L$ for the period is practical because $L$ will be a length of a violin string in Sec. 12.2, of a rod in heat conduction in Sec. 12.5, and so on.

- The transition from period $2\pi$ to be period $p=2L$ is effected by a suitable change of scale, as follows.

- Let $f(x)$ have period $p = 2L$. Then we can introduce a new variable $v$ such that $f(x)$, as a function of $v$, has period $2\pi$. 
Transition

- If we set

\[(1) \quad (a) \quad x = \frac{p}{2\pi}v \quad \text{so that} \quad (b) \quad v = \frac{2\pi}{p}x = \frac{\pi}{L}x\]

then \(v = \pm \pi\) corresponds to \(x = \pm L\). This means that \(f\), as a function of \(v\), has period \(2\pi\) and, therefore, a Fourier series of the form

\[(2) \quad f(x) = f\left(\frac{L}{\pi}v\right) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)\]

with coefficients obtained from (6) in the last section

\[
\begin{align*}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) dv \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) \cos nvdv \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) \sin nvdv
\end{align*}
\]
Transition

- We could use these formulas directly, but the change to $x$ simplifies calculations. Since

\[ v = \frac{\pi}{L} x \]

we have

\[ dv = \frac{\pi}{L} dx \]

and we integrate over $x$ from $-L$ to $L$. Consequently, we obtain for a function $f(x)$ of period $2L$ the Fourier series

\[ f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \]

with the **Fourier coefficients** of $f(x)$ given by the **Euler formulas** ($\pi/L$ in $dx$ cancels $1/\pi$ in (3))
Transition

- Just as in Sec. 11.1, we continue to call (5) with any coefficients a **trigonometric series**. And we can integrate from 0 to $2L$ or over any other interval of length $p=2L$.

\[
\begin{align*}
(0) \quad a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \\
(6) &\quad (a) \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \\
&\quad (b) \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx
\end{align*}
\]
Example 1

- Find the Fourier series of the function (Fig. 263)

\[ f(x) = \begin{cases} 
0 & \text{if } -2 < x < -1 \\
k & \text{if } -1 < x < 1 \\
0 & \text{if } 1 < x < 2 
\end{cases} \]

- Solution. From (6.0) we obtain \( a_0 = k/2 \). From (6a) we obtain

\[ a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos \frac{n\pi x}{2} \, dx = \frac{1}{2} \int_{-1}^{1} k \cos \frac{n\pi x}{2} \, dx = \frac{2k}{n\pi} \sin \frac{n\pi}{2} \]

Thus \( a_n = 0 \) if \( n \) is even and

\[ a_n = \frac{2k}{n\pi} \quad \text{if } n = 1, 5, 9, \ldots \]
\[ a_n = -\frac{2k}{n\pi} \quad \text{if } n = 3, 7, 11, \ldots \]

(6a) \[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx \]
Example 1

- From (6b) we find that $b_n = 0$ for $n = 1, 2, \ldots$. Hence the Fourier series is a **Fourier cosine series** (that is, it has no sine terms)

$$f(x) = \frac{k}{2} + \frac{2k}{\pi} \left( \cos \frac{\pi}{2}x - \frac{1}{3} \cos \frac{3\pi}{2}x + \frac{1}{5} \cos \frac{5\pi}{2}x - + \cdots \right)$$

$$a_n = \frac{2k}{n\pi} \quad \text{if } n = 1, 5, 9, \ldots$$

$$a_n = -\frac{2k}{n\pi} \quad \text{if } n = 3, 7, 11, \ldots$$

(6b) $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$
Example 2

- Find the Fourier series of the function (Fig. 264)

\[
f(x) = \begin{cases} 
-k & \text{if } -2 < x < 0 \\
\frac{k}{2} & \text{if } 0 < x < 2
\end{cases}
\]

\[\text{p=2L=4, } L=2\]

- Solution. Since \(L=2\), we have in (3) \(v = \pi x/2\) and obtain from (8) in Sec. 11.1 with \(v\) instead of \(x\), that is,

\[g(v) = \frac{4k}{\pi} \left( \sin v + \frac{1}{3} \sin 3v + \frac{1}{5} \sin 5v + \cdots \right)\]

the present Fourier series

\[f(x) = \frac{4k}{\pi} \left( \sin \frac{\pi}{2} x + \frac{1}{3} \sin \frac{3\pi}{2} x + \frac{1}{5} \sin \frac{5\pi}{2} x + \cdots \right)\]

Conform this by using (6) and integrating.

(8) in Sec. 11.1 \[\frac{4k}{\pi} (\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots)\]
Example 3

• A sinusoidal voltage $E \sin \omega t$, where $t$ is time, is passed through a half-wave rectifier that clips the negative portion of the wave (Fig. 265). Find the Fourier series of the resulting periodic function

$$u(x) = \begin{cases} 
0 & \text{if } -L < t < 0 \\
E \sin \omega t & \text{if } 0 < t < L
\end{cases}$$

$$p = 2L = \frac{2\pi}{\omega}, L = \frac{\pi}{\omega}$$

• Solution. Since $u=0$ when $-L < t < 0$, we obtain from (6.0), with $t$ instead of $x$,

$$a_0 = \frac{\omega}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} E \sin \omega t dt = \frac{E}{\pi}$$

$$\text{(6.0) } a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx$$
Example 3

- From (6a), by using formula (11) in App. A3.1 with $x = \omega t$ and $y = n\omega t$,

$$a_n = \frac{\omega}{2\pi} \int_0^{\pi/\omega} E \sin \omega t \cos n\omega t dt$$

$$= \frac{\omega E}{2\pi} \int_0^{\pi/\omega} [\sin(1 + n)\omega t + \sin(1 - n)\omega t] dt$$

- If $n=1$, the integral on the right is zero, and if $n=2, 3, \ldots$, we readily obtain

$$a_n = \frac{\omega E}{2\pi} \left[ -\frac{\cos(1 + n)\omega t}{(1 + n)\omega} - \frac{\cos(1 - n)\omega t}{(1 - n)\omega} \right]_0^{\pi/\omega}$$

$$= \frac{E}{2\pi} \left( \frac{-\cos(1 + n)\pi + 1}{1 + n} + \frac{-\cos(1 - n)\pi + 1}{1 - n} \right)$$
Example 3

- If \( n \) is odd, this is equal to zero, and for even \( n \) we have

\[
a_n = \frac{E}{2\pi} \left( \frac{2}{1+n} + \frac{2}{1-n} \right) = -\frac{2E}{(n-1)(n+1)\pi}
\]

- In a similar fashion we find from (6b) that \( b_1 = E/2 \) and \( b_n = 0 \) for \( n = 2, 3, \ldots \). Consequently,

\[
u(t) = \frac{E}{\pi} + \frac{E}{2} \sin \omega t - \frac{2E}{\pi} \left( \frac{1}{1 \cdot 3} \cos 2\omega t + \frac{1}{3 \cdot 5} \cos 4\omega t + \cdots \right)
\]

\[
a_n = \frac{\omega E}{2\pi} \left[ -\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega} \right]_0^{\pi/\omega}
\]

\[
= \frac{E}{2\pi} \left( -\frac{\cos(1+n)\pi}{1+n} + 1 + \frac{-\cos(1-n)\pi}{1-n} + 1 \right)
\]
Simplifications

- If \( f(x) \) is an even function, that is, \( f(-x) = f(x) \) (see Fig. 266), its Fourier series (5) reduces to a Fourier cosine series

\[
(5^*) \quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x
\]

with coefficients (note: integration from 0 to \( L \) only!)

\[
(6^*) \quad a_0 = \frac{1}{L} \int_{0}^{L} f(x) dx \quad \quad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi}{L} x dx
\]

[Fig. 266. Even function]
Simplifications

- If \( f(x) \) is an **odd function**, that is, \( f(-x) = -f(x) \) (see Fig. 267), its Fourier series (5) reduces to a **Fourier sine series**

\[
(5**) \quad f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x
\]

with coefficients

\[
(6**) \quad b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx
\]
Simplifications

- These formulas follow from (5) and (6) by remembering from calculus that the definite integral gives the net area (=area above the axis minus area below the axis) under the curve of a function between the limits of integration. This implies

\[
\begin{align*}
(7) & \quad \int_{-L}^{L} g(x) \, dx = 2 \int_{0}^{L} g(x) \, dx \quad \text{for even } g \\
& \quad \int_{-L}^{L} h(x) \, dx = 0 \quad \text{for odd } h \\
(0) & \quad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \\
(6) & \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx \\
& \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx
\end{align*}
\]
Simplifications

- Formula (7b) implies the reduction to the cosine series (even $f$ makes $f(x) \sin(n\pi x/L)$ odd since $\sin$ is odd) and to the sine series (odd $f$ makes $f(x) \cos(n\pi x/L)$ odd since $\cos$ is even).

- Similar, (7a) reduces the integrals in (6*) and (6**) to integrals from 0 to $L$. These reductions are obvious from the graphs of an even and an odd function.

\[
\begin{align*}
(7) & \quad (a) \quad \int_{-L}^{L} g(x)dx = 2 \int_{0}^{L} g(x)dx \quad \text{for even } g \\
& \quad (b) \quad \int_{-L}^{L} h(x)dx = 0 \quad \text{for odd } h
\end{align*}
\]

\[
\begin{align*}
(6) & \quad (a) \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}dx \\
& \quad (b) \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}dx
\end{align*}
\]

\[
\begin{align*}
(0) & \quad a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx
\end{align*}
\]
Summary

Even Function of Period $2\pi$. If $f$ is even and $L = \pi$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

with coefficients

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) \, dx, \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx, \quad n = 1, 2, \cdots$$

Odd Function of Period $2\pi$. If $f$ is odd and $L = \pi$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

with coefficients

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \cdots$$
Example 4

- The rectangular wave in Example 1 is even. Hence it follows without calculation that its Fourier series is a Fourier cosine series, the $b_n$ are all zero.
- Similarly, it follows that the Fourier series of the odd function in Example 2 is a Fourier sine series.
- In Example 3 you can see that the Fourier cosine series represents $u(t) = E/\pi - \frac{1}{2}E\sin \omega t$. This is an even function.
Theorem 1

- **Sum and Scalar Multiple**
- The Fourier coefficients of a sum $f_1+f_2$ are the sums of the corresponding Fourier coefficients of $f_1$ and $f_2$.
- The Fourier coefficients of $cf$ are $c$ times the corresponding Fourier coefficients of $f$. 
Example 5

- Find the Fourier series of the function (Fig. 268)
  \[ f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi \quad f(x + 2\pi) = f(x) \]

- Solution. We have \( f = f_1 + f_2 \), where \( f_1 = x, f_2 = \pi \). The Fourier coefficients of \( f_2 \) are zero, except for the first one (the constant term), which is \( \pi \). Hence, by Theorem 1, the Fourier coefficients \( a_n, b_n \) are those of \( f_1 \), except for \( a_0 \), which is \( \pi \). Since \( f_1 \) is odd, \( a_n=0 \) for \( n=1, 2, \ldots, \) and

  \[ b_n = \frac{2}{\pi} \int_0^{\pi} f_1(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx \]
Example 5

- Integrating by parts, we obtain

\[ b_n = \frac{2}{\pi} \left[ -\frac{x \cos n x}{n} \right]_0^\pi + \frac{1}{n} \int_0^\pi \cos n x \, dx \] = -\frac{2}{n} \cos n \pi

Hence, \( b_1 = 2, b_2 = -\frac{2}{2}, b_3 = \frac{2}{3}, b_4 = -\frac{2}{4}, \ldots, \) and the Fourier series of \( f(x) \) is

\[ f(x) = \pi + 2 \left( \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \cdots \right) \]

Fig. 269. Partial sums \( S_1, S_2, S_3, S_{20} \) in Example 5
Half-Range Expansions

- Half-range expansions are Fourier series. We want to represent \( f(x) \) in Fig. 270.0 by a Fourier series, where \( f(x) \) may be the shape of a distorted violin string or the temperature in a metal bar of length \( L \), for example.

Fig. 270. Even and odd extensions of period \( 2L \)
Half-Range Expansions

- We would extend $f(x)$ as a function of period $L$ and develop the extended function into a Fourier series. But this series would, in general, contain both cosine and sine terms. We can do better and get simpler series. Indeed, for our given $f$ we can calculate Fourier coefficients from (6*) or form (6**). And we have a choice and can make what seems more practical.

\[(6*) \quad a_0 = \frac{1}{L} \int_0^L f(x)dx \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}dx\]

\[(6**) \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}dx\]
Half-Range Expansions

- If we use (6*), we get (5*). This is the **even periodic extension** $f_1$ of $f$ in Fig. 270a. If we use (6**), we get (5**). This is the **odd periodic extension** $f_2$ of $f$ in Fig. 270b.

- Both extensions have period $2L$. This motivates the name **half-range expansions**: $f$ is given only on half the range, that is, on half the interval of periodicity of length $2L$. 

Example 6

- Find the two half-range expansion of the function (Fig. 271)

\[ f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L - x) & \text{if } \frac{L}{2} < x < L \end{cases} \]

- Solution. (a) **Even periodic extension.** From (6*) we obtain

\[
a_0 = \frac{1}{L} \left[ \frac{2k}{L} \int_0^{L/2} x \, dx + \frac{2k}{L} \int_{L/2}^{L} (L - x) \, dx \right] = \frac{k}{2}
\]

\[
a_n = \frac{2}{L} \left[ \frac{2k}{L} \int_0^{L/2} x \cos \frac{n\pi}{L} \, dx + \frac{2k}{L} \int_{L/2}^{L} (L - x) \cos \frac{n\pi}{L} \, dx \right]
\]

We consider \( a_n \). For the first integral we obtain by integration by parts

\[
\int_0^{L/2} x \cos \frac{n\pi}{L} \, dx = \frac{Lx}{n\pi} \sin \frac{n\pi}{L} \, x \Big|_0^{L/2} - \frac{L}{n\pi} \int_0^{L/2} \sin \frac{n\pi}{L} \, x \, dx
\]

\[
= \frac{L^2}{2n\pi} \sin \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} (\cos \frac{n\pi}{2} - 1)
\]
\[ \int uv' \, dx = uv - \int u'v \, dx \]

**Example 6**

- Similarly, for the second integral we obtain
  \[ \int_{L/2}^{L} (L - x) \cos \frac{n\pi}{L} x \, dx = \frac{L}{n\pi} (L - x) \sin \frac{n\pi}{L} x \bigg|_{L/2}^{L} + \frac{L}{n\pi} \int_{L/2}^{L} \sin \frac{n\pi}{L} x \, dx \]
  \[ = (0 - \frac{L}{n\pi} (L - \frac{L}{2}) \sin \frac{n\pi}{2}) - \frac{L^2}{n^2\pi^2} (\cos n\pi - \cos \frac{n\pi}{2}) \]

- We insert these two results into the formula for \( a_n \). The sine terms cancel and so does a factor \( L^2 \). This gives
  \[ a_n = \frac{4k}{n^2\pi^2} (2 \cos \frac{n\pi}{2} - \cos n\pi - 1) \]

  Thus, \( a_2 = -16k/(2^2\pi^2) \), \( a_6 = -16k/(6^2\pi^2) \), \( a_{10} = -16k/(10^2\pi^2) \), ...

  and \( a_n = 0 \) if \( n \neq 2, 6, 10, 14, ... \). Hence the first half-range expansion of \( f(x) \) is Fig. (272a)

  \[ f(x) = \frac{k}{2} - \frac{16k}{\pi^2} \left( \frac{1}{2^2} \cos \frac{2\pi}{L} x + \frac{1}{6^2} \cos \frac{6\pi}{L} x + \cdots \right) \]

  This Fourier cosine series represents the even periodic extension of the given function \( f(x) \), of period \( 2L \).
Example 6

• (b) **Odd periodic extension.** Similarly, from (6**) we obtain
  \[ b_n = \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2} \]
  Hence the other half-range expansion of \( f(x) \) is (Fig. 272b)
  \[
  f(x) = \frac{8k}{\pi^2} \left( \frac{1}{1^2} \sin \frac{\pi}{L} x - \frac{1}{3^2} \sin \frac{3\pi}{L} x + \frac{1}{5^2} \sin \frac{5\pi}{L} x - + \cdots \right)
  \]
  The series represents the odd periodic extension of \( f(x) \), of period \( 2L \).

Fig. 272. Periodic extensions of \( f(x) \) in Example 6
11.3 Forced Oscillations
Forced Oscillations

- Fourier series have important application for both ODEs and PDEs.
- From Sec. 2.8 we know that forced oscillations of a body of mass \( m \) on a spring of modulus \( k \) are governed by the ODE

\[
my'' + cy' + ky = r(t)
\]

where \( y = y(t) \) is the displacement from rest, \( c \) the damping constant, \( k \) the spring constant, and \( r(t) \) the external force depending on time \( t \).
Forced Oscillations

- If $r(t)$ is a sine or cosine function and if there is damping ($c>0$), then the steady-state solution is a harmonic oscillation with frequency equal to that of $r(t)$.

- However, if $r(t)$ is not a pure sine or cosine function but is any other periodic function, then the steady-state solution will be a superposition of harmonic oscillations with frequencies equal to that of $r(t)$ and integer multiplies of these frequencies.
Forced Oscillations

- And if one of these frequencies is close to the (practical) resonant frequency of the vibrating system (see Sec. 2.8), then the corresponding oscillation may be the dominant part of the response of the system to the external force.

- This is what the use of Fourier series will show us. Of course, this is quite surprising to an observer unfamiliar with Fourier series, which are highly important in the study of vibrating systems and resonance.
Example 1

- In (1), let \( m = 1 \) (g), \( c = 0.05 \) (g/sec), and \( k = 25 \) (g/sec\(^2\)), so that (1) becomes

\[
y'' + 0.05y' + 25y = r(t)
\]

where \( r(t) \) is measured in \( g \cdot \text{cm/sec}^2 \). Let (Fig. 276)

\[
r(t) = \begin{cases} 
  t + \frac{\pi}{2} & \text{if } -\pi < t < 0 \\
  -t + \frac{\pi}{2} & \text{if } 0 < t < \pi 
\end{cases} \quad r(t + 2\pi) = r(t)
\]

Find the steady-state solution \( y(t) \).

![Fig. 276. Force in Example 1](image)
Example 1

- Solution. We represent \( r(t) \) by a Fourier series, finding

\[
(3) \quad r(t) = \frac{4}{\pi} \left( \cos t + \frac{1}{3^2} \cos 3t + \frac{1}{5^2} \cos 5t + \cdots \right)
\]

Then we consider the ODE

\[
(4) \quad y'' + 0.05y' + 25y = \frac{4}{n^2 \pi} \cos nt \quad (n = 1, 3, \ldots)
\]

whose right side is a single term of the series (3). From Sec. 2.8 we know that the steady-state solution \( y_n(t) \) of (4) is of the form

\[
(5) \quad y_n = A_n \cos nt + B_n \sin nt
\]

By substituting this into (4) we find that

\[
(6) \quad A_n = \frac{4(25 - n^2)}{n^2 \pi D_n} \quad B_n = \frac{0.2}{n \pi D_n} \quad D_n = (25 - n^2)^2 + (0.05n)^2
\]
Example 1

- Since the ODE (2) is linear, we may expect the steady-state solution to be

\[ y = y_1 + y_3 + y_5 + \cdots \]

where \( y_n \) is given by (5) and (6). In fact, this follows readily by substituting (7) into (2) and using the Fourier series of \( r(t) \), provided the termwise differentiation of (7) is permissible.

\[
(5) \quad y_n = A_n \cos nt + B_n \sin nt
\]
Example 1

- From (6) we find that the amplitude of (5) is (a factor $\sqrt{D_n}$ cancels out)

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2\pi \sqrt{D_n}}$$

Values of the first few amplitudes are

$C_1 = 0.0531, C_3 = 0.0088, C_5 = 0.2037, C_7 = 0.0011, C_9 = 0.0003$

(5)  $y_n = A_n \cos nt + B_n \sin nt$

(6)  $A_n = \frac{4(25 - n^2)}{n^2\pi D_n}$  $B_n = \frac{0.2}{n\pi D_n}$  $D_n = (25 - n^2)^2 + (0.05n)^2$
Example 1

- Figure 277 shows the input (multiplied by 0.1) and the output. For $n=5$ the quantity $D_n$ is very small, the denominator of $C_5$ is small, and $C_5$ is so large that $y_5$ is the dominating term in (7).

- Hence the output is almost a harmonic oscillation of five times the frequency of the driving force, a little distorted due to the term $y_1$, whose amplitude is about 25% of that of $y_5$. You could make the situation still more extreme by decreasing the damping constant $c$. 

*Fig. 277.* Input and steady-state output in Example 1