Lecture 6 Fourier Analysis

Fundamentals of Digital Signal Processing
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Fourier Series

- We can synthesize periodic waveforms by using a sum of harmonically related sinusoids.
- Now we want to show a general theory (the mathematical theory of *Fourier series*)

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \]

- \( T_0 \) is the fundamental period of the periodic signal \( x(t) \)
- The \( k \)th complex exponential has a frequency equal to \( f_k = k/T_0 \) Hz, so all the frequencies are integer multiples of the fundamental frequency \( f_0 = 1/T_0 \) Hz.
Fourier Series

- Two aspects of the Fourier theory: analysis and synthesis
- Start from $x(t)$ and calculating $\{a_k\}$ is called Fourier analysis.
- The reverse process of starting from $\{a_k\}$ and generating $x(t)$ is called Fourier synthesis.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos((2\pi/T_0)kt + \phi_k)$$

- By clever choice of $a_k$, we can represent a number of interesting periodic waveforms.

$$a_k = \begin{cases} 
A_0 & \text{for } k = 0 \\
\frac{1}{2} A_k e^{j\phi_k} & \text{for } k \neq 0
\end{cases}$$

$$a_{-k} = a_k^*$$
Fourier Series: Analysis

• How do we go from $x(t)$ to $a_k$?
  • *Fourier series integral*

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt} \, dt \quad T_0 \text{ is the fundamental period}$$

• DC component: $a_0$ is simply the average value of the signal over one period

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) \, dt$$

• The Fourier integral is convenient if we have a formula that defines $x(t)$ over one period.
Fourier Series Derivation

- Integral of a complex exponential over an integral number of periods is zero
  \[\int_0^{T_0} e^{j(2\pi/T_0)kt} \, dt = \frac{e^{j(2\pi/T_0)kT_0}}{j(2\pi/T_0)k} \bigg|_0^{T_0} = \frac{e^{j(2\pi/T_0)kT_0} - 1}{j(2\pi/T_0)k} = 0\]

- The numerator is zero because \(e^{j2\pi k} = 1\) for any integer \(k\) (positive or negative)

- Prove by using Euler’s formula
  \[\int_0^{T_0} e^{j(2\pi/T_0)kt} \, dt = \int_0^{T_0} \cos((2\pi/T_0)kt) \, dt + j \int_0^{T_0} \sin((2\pi/T_0)kt) \, dt = 0\]
Fourier Series Derivation

- Define $v_k(t)$ to be the complex exponential of frequency $\omega_k = (2\pi / T_0)k$, then $v_k(t) = e^{j(2\pi / T_0)kt}$

$$
v_k(t + T_0) = e^{j(2\pi / T_0)k(t + T_0)}
= e^{j(2\pi / T_0)kt} e^{j(2\pi / T_0)kT_0}
= e^{j(2\pi / T_0)kt} e^{j2\pi k}
= e^{j(2\pi / T_0)kt}
= v_k(t)
$$

$e^{j2\pi k} = 1$
Fourier Series Derivation

- Generalize the zero-integral property of the complex exponential to involve two signals

\[
\int_{0}^{T_0} v_k(t)v_\ell^*(t)\,dt = \begin{cases} 
0 & \text{if } k \neq \ell \\
T_0 & \text{if } k = \ell 
\end{cases}
\]

Orthogonal Property

\[
\int_{0}^{T_0} v_k(t)v_\ell^*(t)\,dt = \int_{0}^{T_0} e^{j(2\pi/T_0)kt}e^{-j(2\pi/T_0)\ell t}\,dt \\
= \int_{0}^{T_0} e^{j2\pi(T_0/k)(k-\ell)t}\,dt
\]

\(v_\ell^*\) denotes the complex conjugate

The inner product of \(v_k(t)\) and \(v_\ell(t)\)
Fourier Series Derivation

- When \( k = \ell \), the integral is
  \[
  \int_{0}^{T_0} e^{j(2\pi/T_0)(k-\ell)t} dt = \int_{0}^{T_0} e^{j\ell t} dt = \int_{0}^{T_0} 1 dt = T_0
  \]

- When \( k \neq \ell \)
  \[
  \int_{0}^{T_0} e^{j(2\pi/T_0)(k-\ell)t} dt = \int_{0}^{T_0} e^{j(2\pi/T_0)m t} dt = 0
  \]
  \[m = k - \ell \neq 0\]
  \[
  \int_{0}^{T_0} e^{j(2\pi/T_0)k t} dt = 0
  \]
Fourier Series Derivation

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \]

- We can multiply both sides by the complex exponential \( v^*_\ell(t) \) and integrate over the period \( T_0 \)
  \[
  \int_{0}^{T_0} x(t)e^{-j(2\pi/T_0)\ell t} \, dt
  = \int_{0}^{T_0} \left( \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \right) e^{-j(2\pi/T_0)\ell t} \, dt
  = \sum_{k=-\infty}^{\infty} a_k \left( \int_{0}^{T_0} e^{j(2\pi/T_0)(k-\ell)t} \, dt \right) = a_\ell T_0
  
  There is a value only when \( k = \ell \)

- We can isolate one complex amplitude (\( a_\ell \)) in the final step by applying the orthogonality property.
Fourier Series Derivation

- The final analysis formula is obtained by dividing both sides by $T_0$ to get $a_\ell$.

**Fourier Analysis Equation**

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} \, dt$$

**Fourier Synthesis Equation**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$
Spectrum of the Fourier Series

- The Fourier series coefficients $a_k$ are the complex amplitudes that define the spectrum of $x(t)$.
- E.g. Consider the coefficients for $x(t) = \sin^3(3\pi t)$ and sketch its spectrum.
- Two methods: (1) plug $x(t)$ into the Fourier integral, or (2) use the inverse Euler formula to expand $x(t)$ into a sum of complex exponentials.
Spectrum of the Fourier Series

\[ x(t) = \left( \frac{e^{j3\pi t} - e^{-j3\pi t}}{2j} \right)^3 \]

\[ = \frac{1}{-8j} (e^{j9\pi t} - 3e^{j6\pi t}e^{-j3\pi t} + 3e^{j3\pi t}e^{-j6\pi t} - e^{-j9\pi t}) \]

\[ = \frac{j}{8}e^{j9\pi t} + \frac{-3j}{8}e^{j3\pi t} + \frac{3j}{8}e^{-j3\pi t} + \frac{-j}{8}e^{-j9\pi t} \]

- Contains four frequencies: \( \omega = \pm 3\pi \) and \( \omega = \pm 9\pi \)
- The fundamental frequency is \( \omega_0 = 3\pi \) rad/sec
- The Fourier series coefficients are indexed in terms of the fundamental frequency
- It’s not necessary to evaluate an integral to obtain the \( \{a_k\} \) coefficients

\[
 a_k = \begin{cases} 
 0 & \text{for } k = 0 \\
 \mp j^{3/8} & \text{for } k = \pm 1 \\
 0 & \text{for } k = \pm 2 \\
 \pm j^{1/8} & \text{for } k = \pm 3 \\
 0 & \text{for } k = \pm 4, \pm 5, \pm 6, \ldots 
\end{cases}
\]
Spectrum of the Fourier Series

- We have four nonzero $a_k$ components located at the four frequencies: $\omega = \{-9\pi, -3\pi, 3\pi, 9\pi\}$ rad/sec
- We prefer to plot the spectrum versus frequency in hertz, so the spectrum lines are at $f = -4.5, -1.5, 1.5, 4.5$ Hz

$$a_k = \begin{cases} 
0 & \text{for } k = 0 \\
\mp j\frac{3}{8} & \text{for } k = \pm 1 \\
0 & \text{for } k = \pm 2 \\
\pm j\frac{1}{8} & \text{for } k = \pm 3 \\
0 & \text{for } k = \pm 4, \pm 5, \pm 6, \ldots
\end{cases}$$

$$e^{-j\pi/2} = -j$$
EXERCISE 3.4:

Use the Fourier integral to determine all the Fourier Series coefficients of the “sine-cubed” signal. In other words, evaluate the integral

$$a_k = \frac{1}{T_0} \int_0^{T_0} \sin^3(3\pi t)e^{-j(2\pi/T_0)k\pi t} \, dt$$

for all $k$.

Hints: find the period first, so that the integration interval is known. In addition, you might find it easier to convert the $\sin^3(\cdot)$ function to exponential form (via the inverse Euler formula for $\sin(\cdot)$) before doing the Fourier integral on each of four different terms. If you then invoke the orthogonality property on each integral, you should get exactly the same answer as (3.29).
Exercise 3.4

\[ a_k = \frac{1}{T_0} \int_0^{T_0} \sin^3(3\pi t) e^{-j(2\pi / T_0)kt} \, dt \]

The period of \( \sin^3(3\pi t) \) is \( T_0 = \frac{2}{3} \)

Use the identity \((a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\)

to expand \( \sin^3(3\pi t) \)

\[ \sin(3\pi t) = \frac{e^{j3\pi t} - e^{-j3\pi t}}{2j} \]

\[ \sin^3(3\pi t) = \frac{1}{(2j)^3} \left( e^{j9\pi t} - 3e^{j6\pi t} - e^{-j3\pi t} + 3e^{j3\pi t} - e^{-j6\pi t} - e^{-j9\pi t} \right) \]

\[ = \frac{j}{8} e^{j9\pi t} - \frac{3j}{8} e^{j6\pi t} + \frac{3j}{8} e^{-j3\pi t} - \frac{j}{8} e^{-j6\pi t} - \frac{j}{8} e^{-j9\pi t} \]
Exercise 3.4

Integrate each term individually. For example,

\[ \frac{1}{T_0} \int_{0}^{T_0} \frac{j}{8} e^{i(\omega_0 t)} e^{-j(\omega f_0)kt} \, dt \]

\[ = \frac{3j}{2} \int_{0}^{T_0} \frac{j}{8} e^{i(\omega_0 t)} e^{-j(3\pi k) t} \, dt \]

\[ = \frac{3j}{16} \int_{0}^{T_0} e^{j3\pi(3-k)t} \, dt = \begin{cases} \frac{3j}{16} \cdot \frac{2}{3} & \text{if } k=3 \\ \text{integrate if } k \neq 3 \end{cases} \]

\[ = \frac{3j}{16} \left. \frac{e^{j3\pi(3-k)t}}{j3\pi(3-k)} \right|_{0}^{3/2} \]

\[ = \frac{1}{16(3-k)} \left[ e^{j2\pi(3-k)} - e^{j0} \right] = 0 \]

\[ \Rightarrow a_3 = \frac{j}{8} \]

Likewise, we get: \( a_1 = -\frac{3j}{8}, a_{-1} = \frac{3j}{8}, a_3 = -\frac{j}{8} \)
Fourier Analysis of Periodic Signals

\[ x_N(t) = \sum_{k=-N}^{N} a_k e^{j(2\pi/T_0)kt} \]

where

- \( a_k \) are the Fourier coefficients.
- \( T_0 \) is the period.
- \( f_0 = \frac{1}{T_0} \) is the fundamental frequency.


DSP, CSIE, CCU
The Square Wave

\[ s(t) = \begin{cases} 
1 & \text{for } 0 \leq t < \frac{1}{2}T_0 \\
0 & \text{for } \frac{1}{2}T_0 \leq t < T_0 
\end{cases} \]

- We will derive a formula that depends on \( k \) for the complex amplitudes \( a_k \).
- We substitute the definition of \( x(t) \) into the integral

\[
a_k = \frac{1}{T_0} \int_{0}^{\frac{1}{2}T_0} (1) e^{-j(2\pi/T_0)kt} \, dt
\]

\[
a_k = \frac{1}{T_0} \int_{0}^{T_0} x(t) e^{-j(2\pi/T_0)kt} \, dt
\]
The Square Wave

\[ a_k = \frac{1}{T_0} \int_0^{1T_0} (1) e^{-j(2\pi/T_0)kt} \, dt \]

\[ = \left( \frac{1}{T_0} \right) \left[ \frac{e^{-j(2\pi/T_0)kt}}{-j(2\pi/T_0)k} \right]_0^{1T_0} \]

\[ = \left( \frac{1}{T_0} \right) \frac{e^{-j(2\pi/T_0)k(1T_0)} - e^{-j(2\pi/T_0)k(0)}}{-j(2\pi/T_0)k} \]

\[ = \frac{e^{-j\pi k} - 1}{-j2\pi k} \]

\[ e^{-j\pi} = -1 \]

\[ a_k = \frac{e^{-j\pi k} - 1}{-j2\pi k} = \frac{1 - (-1)^k}{j2\pi k} \]

for \( k \neq 0 \)

\[ a_0 = \frac{1}{T_0} \int_0^{1T_0} (1) e^{-j(0)kt} \, dt \]

\[ = \frac{1}{T_0} \int_0^{1T_0} (1) dt = \frac{1}{T_0} \left( \frac{1}{2}T_0 \right) = \frac{1}{2} \]
The Square Wave

\[ a_k = \begin{cases} 
\frac{1}{j\pi k} & k = \pm 1, \pm 3, \pm 5, \ldots \\
0 & k = \pm 2, \pm 4, \pm 6, \ldots \\
\frac{1}{2} & k = 0 
\end{cases} \]

- The magnitude of these coefficients decreases as \( k \to \infty \), so the high frequency terms contribute less when synthesizing the waveform.
DC Value of a Square Wave

- The Fourier series coefficient for $k = 0$ has a special interpretation as the *average value* of the signal $x(t)$.

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) \, dt$$

- The integral is the area under the function $x(t)$ for one period.

- In the square wave, the average is $\frac{1}{2}$ because the signal is equal to $+1$ for half the period and then $0$ for the other half.

- In the synthesis formula, the $a_0$ coefficient is an additive constant, which will move the plot of signal up or down vertically.

- DC – direct current, a constant value of current
Spectrum for a Square Wave

- Spectrum of the square wave when the fundamental frequency is 25 Hz
- The magnitude of these coefficients drops off as $1/k$.

$$a_k = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \pm 5, \ldots \\ 0 & k = \pm 2, \pm 4, \pm 6, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

$$\frac{1}{j} = -j$$
Synthesis of a Square Wave

\[ x_N(t) = \sum_{k=-N}^{N} a_k e^{j(2\pi/T_0)kt} \]

- \( N = 3, 7, \) and 17
- As \( N \) increases, the sum of cosines appears to converge to the constant values 1 and 0, but the convergence is not uniformly good.
- The “ears” at the discontinuous steps never go away completely – \textit{Gibbs phenomenon}
Synthesis of a Square Wave

\[ x_3(t) = a_{-3}e^{-j3\omega_0t} + a_{-1}e^{-j\omega_0t} + a_0 + a_1e^{j\omega_0t} + a_3e^{j3\omega_0t} \]

\[ = \frac{1}{-j3\pi}e^{-j3\omega_0t} + \frac{1}{-j\pi}e^{-j\omega_0t} + a_0 + \frac{1}{j\pi}a_1e^{j\omega_0t} + \frac{1}{j3\pi}a_3e^{j3\omega_0t} \]

\[ = a_0 + \frac{1}{\pi}(e^{-j\pi/2}e^{j\omega_0t} + e^{j\pi/2}e^{-j\omega_0t}) + \frac{1}{3\pi}(e^{-j\pi/2}e^{j3\omega_0t} + e^{j\pi/2}e^{-j3\omega_0t}) \]

\[ = \frac{1}{2} + \frac{2}{\pi}\cos(\omega_0t - \pi/2) + \frac{2}{3\pi}\cos(3\omega_0t - \pi/2) \]
Triangle Wave

\[ x(t) = \begin{cases} 
2t/T_0 & \text{for } 0 \leq t < \frac{1}{2}T_0 \\
2(T_0 - t)/T_0 & \text{for } \frac{1}{2}T_0 \leq t < T_0 
\end{cases} \]

\[ a_0 = \frac{1}{T_0} \int_0^{T_0} x(t)dt \]

- If we recognize that the integral over one period is the area under the triangle, we get

\[ a_0 = \frac{1}{T_0} \text{(area)} = \frac{1}{T_0}(T_0)(\frac{1}{2}) = \frac{1}{2} \]
Triangle Wave

- For $k \neq 0$, we must break the Fourier series analysis integral into two sections.

$$a_k = \frac{1}{T_0} \int_0^{\frac{1}{2}T_0} (2t/T_0) e^{-j(2\pi/T_0)kt} dt +$$

$$\frac{1}{T_0} \int_{\frac{1}{2}T_0}^{T_0} (2(T_0 - t)/T_0) e^{-j(2\pi/T_0)kt} dt$$

$$a_k = \frac{e^{-jk\pi} - 1}{\pi^2 k^2}$$

$$e^{-jk\pi} = (-1)^k$$

$$a_k = \begin{cases} 
\frac{-2}{\pi^2 k^2} & k = \pm 1, \pm 3, \pm 5, \ldots \\
0 & k = \pm 2, \pm 4, \pm 6, \ldots \\
\frac{1}{2} & k = 0 
\end{cases}$$
EXERCISE 3.7:
Make a plot of the spectrum for the triangle wave (similar to Fig. 3-16 for the square wave). Use the complex amplitudes from (3.39) and assume that \( f_0 = 25 \) Hz.

\[
a_k = \begin{cases} 
\frac{1}{2} & k = 0 \\
0 & k = \pm 2, \pm 4, \pm 6, \ldots \\
\frac{-2}{\pi k^2} & k = \pm 1, \pm 3, \pm 5, \ldots 
\end{cases}
\]

\[f_0 = 25 \text{ Hz} \implies \omega_0 = 2\pi f_0 = 50\pi \text{ rad/s}\]
Synthesis of a Triangle Wave

- Unlike the square wave, the triangle wave is a continuous signal. Therefore, it’s easier to approximate with a finite Fourier sum.
- Two cases: $N=3$ and $11$. $f_0 = 25$ Hz. In the $N=11$ case the approximation is nearly indistinguishable from the triangularly-shaped waveform.
Convergence of Fourier Synthesis

- We can think of the finite Fourier sum as making an approximation to the true signal, i.e.

\[ x(t) \approx x_N(t) = \sum_{k=-N}^{N} a_k e^{j(2\pi/T_0)kt} \]

- We might hope that with enough complex exponentials we could make the approximation perfect.

- Define an error signal, \( e_N(t) \), as the difference between the true signal and the synthesis with \( N \) terms.

- Worst-case error

\[ E_{\text{worst}} = \max_{t \in [0, T_0]} |x(t) - x_N(t)| \]
Convergence of Fourier Synthesis

- If we zoom this figure in, these errors is 0.0497 for $N=3$ and 0.0168 for $N=11$. The maximum error decreases to zero as $N \to \infty$.

- It’s the case for the discontinuous square wave where the maximum error is always half the size of the jump in the waveform right at the discontinuity point with an overshoot of about 9% of the size of the discontinuity on either side.
Time-Frequency Spectrum

- We have seen that a wide range of interesting waveforms can be synthesized by the equation
  \[ x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \phi_k) \]
- These waveforms range from constants, to cosine signals, to general periodic signals, to complicated-looking signals that are not periodic.
- One assumption have made so far is that the amplitudes, phases, and frequencies do not change with time.
- However, most real-world signals exhibit frequency changes over time. Music is the best example.
Time-Frequency Spectrum

- For very short time intervals, the music may have a "constant" spectrum, but over the long term, the frequency content of the music changes dramatically.
- Most interesting signals can be modeled as a sum of sinusoids if we let the frequencies, amplitudes, and phases vary with time.
- A way to describe such time-frequency variations – *spectrogram*. (頻譜圖)
Stepped Frequency

- The simplest example of time-varying frequency content is to make a waveform whose frequency stays constant for a short duration and then steps to higher (or lower) frequency
- An octave is doubling the frequency

<table>
<thead>
<tr>
<th>Middle C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>A</th>
<th>B</th>
<th>C</th>
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<tbody>
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<td>262 Hz</td>
<td>294</td>
<td>330</td>
<td>349</td>
<td>392</td>
<td>440</td>
<td>494</td>
<td>523</td>
</tr>
</tbody>
</table>
Spectrogram Analysis

- It’s not easy to write a simple mathematical formula like the Fourier series integral to do the analysis. (Chapter 13)
- `MATLAB specgram` function
  - The calculation is performed by doing a frequency analysis on short segments (e.g. 25.6 msec) of the signal and plotting the results at the specific time at which the analysis is done.
  - By repeating the process with slight displacement in time, a two-dimensional array is created whose magnitude can be displayed as a grayscale image, whose horizontal axis is time and whose vertical axis is frequency.
Demo

PLAY A SCALE

FUR ELISE

BATMAT (MALE SPEAKER)

TRAIN WHISTLE

BEETHOVEN'S FIFTH (Glory by GRiFFIN)
Frequency Modulation

- Create signals whose frequency is time-varying
- A “chirp” signal is a swept-frequency signal whose frequency changes linearly from some low value to a high one.
  - E.g. begin at 220 Hz and go up to 2320 Hz
- One method for producing such a signal is to concatenate a large number of short constant-frequency sinusoids
  - Boundary between the short sinusoids will be discontinuous unless we adjust the initial phase of each small sinusoid
Frequency Modulation

- Write a formula to get time-varying frequency
- If we regard a constant-frequency sinusoid as the real part of a complex phasor
  \[ x(t) = \Re \{ Ae^{j(\omega_0 t + \phi)} \} = A \cos(\omega_0 t + \phi) \]
- Then the *angle function* of this signal is the exponent \((\omega_0 t + \phi)\) which obviously changes linearly with time. The time derivative of the angle function is \(\omega_0\), which equals the constant frequency.
- We adopt the following general notation for the class of signals with time-varying angle function:
  \[ x(t) = \Re \{ Ae^{j\psi(t)} \} = A \cos(\psi(t)) \]

\(\psi(t)\) denotes the angle function versus time.
Frequency Modulation

- We can create a signal with quadratic angle function by defining
  \[ \psi(t) = 2\pi \mu t^2 + 2\pi f_0 t + \phi \]

- Now we can define the *instantaneous frequency* for these signals as the slope of the angle function (i.e. its derivative)
  \[ \omega_i(t) = \frac{d}{dt} \psi(t) \quad \omega_i(t) \quad \text{(rad/sec)} \]

- Where the units of \[ \frac{2\pi}{2\pi} \] are rad/sec, or, if we divide by
  \[ f_i(t) = \frac{1}{2\pi} \frac{d}{dt} \psi(t) \quad \text{(Hz)} \]
Frequency Modulation

- If the angle function of \( x(t) \) is quadratic, then its frequency changes linearly with time; that is,
  \[
  f(t) = 2\mu t + f_0
  \]

- The frequency variation produced by the time-varying angle function is called frequency modulation, and the signals of this class are called **FM signals**.

- Since the linear variation of the frequency can produce an audible sound similar to a siren or a chirp, the linear FM signals also called chirp signals.
Frequency Modulation

- The instantaneous frequency is the derivative of the angle function. Thus, if a certain linear frequency sweep is desired, the actual angle function is obtained from the integral of $\omega_i(t)$.

- Suppose we want to synthesize a frequency sweep from $f_1 = 220$ Hz to $f_2 = 2320$ Hz over a 3-sec time interval, i.e. the beginning and ending times are $t=0$ and $t=T_2=3$ sec.

  $$f_i(t) = \frac{f_2 - f_1}{T_2}t + f_1 = \frac{2320 - 220}{3}t + 220$$

- Integrate $2\pi f_i(t)$ to get the angle function:

  $$\psi(t) = \int_0^t \omega_i(u)du = \int_0^t 2\pi\left(\frac{2320 - 220}{3}u + 220\right)du$$

  $$= 700\pi t^2 + 440\pi t + \phi$$

- The phase shift $\phi$ is an arbitrary constant. The chirp signal is $x(t) = \cos(\psi(t))$.
Example

- $f_1 = 100$ Hz, $f_2 = 500$ Hz, $T_2 = 0.04$ sec
- Instantaneous frequency
  - Concentrate on the time range $0.019 \leq t \leq 0.021$
  - the 300-Hz sinusoid matches the chirp in this time region
- See near $t = 0.04s$
  - where the chirp frequency is equal to 500 Hz
Instantaneous Frequency

\[ x(t) = A \cos(\psi(t)) \]

- The instantaneous frequency of the signal is the derivative of the angle function \( \psi(t) \)
- If \( \psi(t) \) is constant, the frequency is zero.
- If \( \psi(t) \) is linear, \( x(t) \) is a sinusoid at some fixed frequency.
- If \( \psi(t) \) is quadratic, \( x(t) \) is a chirp signal whose frequency changes linearly versus time.
- More complicated variations of \( \psi(t) \) can produce a wide variety of signals. One application is in music synthesis.