Lecture 3 Complex Exponential Signals

Fundamentals of Digital Signal Processing

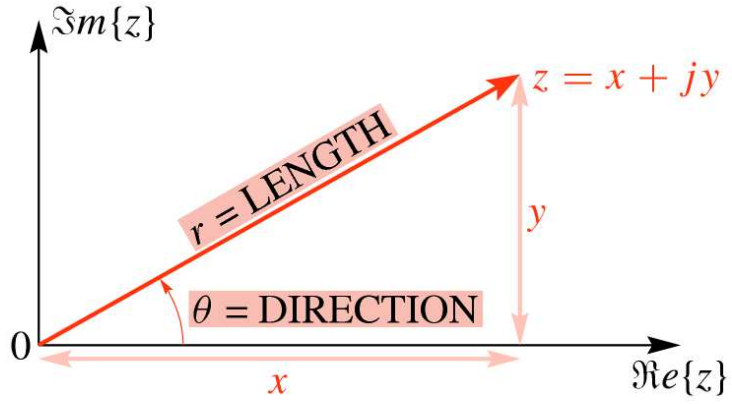
Spring, 2012

Wei-Ta Chu

2012/3/1

Review of Complex Numbers

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = r\cos\theta + jr\sin\theta$$
$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(\frac{y}{x})$$



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- Using Euler's famous formula for the complex exponential

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$z = re^{j\theta} = r\cos\theta + jr\sin\theta$$

- The complex exponential polar form of a complex number is most convenient when calculating a complex multiplication or division. (see Appendix A)

Complex Exponential Signals

- The complex exponential signal is defined as

$$z(t) = Ae^{j(\omega_0 t + \phi)}$$

- It's a complex-valued function of *t*, where the magnitude of *z*(*t*) is |z(*t*)|=*A* and the angle of *z*(*t*) is

$$\arg z(t) = (\omega_0 t + \phi)$$

- Using Euler's formula

$$z(t) = Ae^{j(\omega_0 t + \phi)}$$
$$= A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

- The real part is a real cosine signal as defined previously.

Complex Exponential Signals

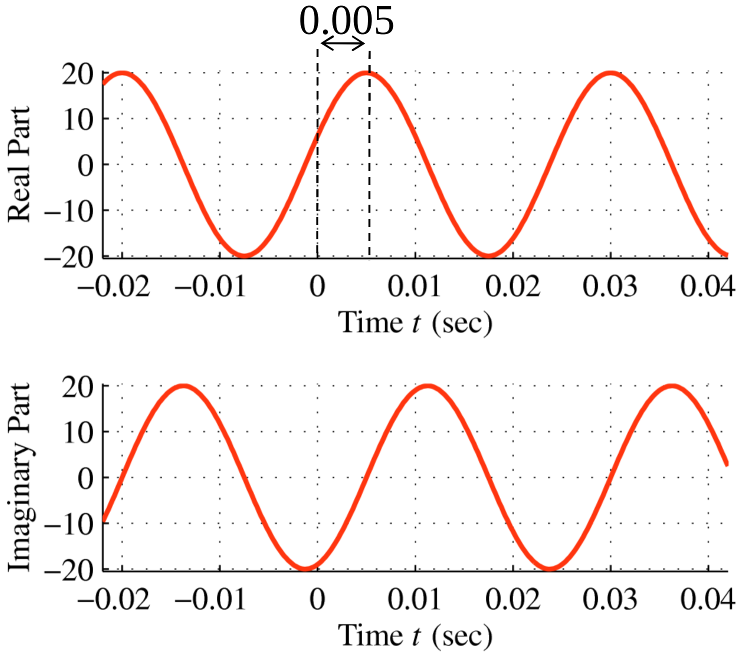
- Example:

$$z(t) = 20e^{j(2\pi(40)t - 0.4\pi)}$$
$$= 20e^{j(80\pi t - 0.4\pi)}$$
$$= 20\cos(80\pi t - 0.4\pi) + j20\sin(80\pi t - 0.4\pi)$$
$$= 20\cos(80\pi t - 0.4\pi) + j20\cos(80\pi t - 0.9\pi)$$

$$\sin(\omega_0 t + \phi) = \cos(\omega_0 t + \phi - \pi/2)$$

$$T_0 = \frac{1}{f_0} = \frac{1}{40} = 0.025 \text{ sec}$$

$$t_1 = -\frac{\phi}{2\pi f_0} = -\frac{-0.4\pi}{2\pi 40} = 0.005 \text{ sec}$$



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Complex Exponential Signals

- The main reason we are interested in the complex exponential signal is that it is an alternative representation for the real cosine signal.

$$x(t) = \mathfrak{Re}\{Ae^{j(\omega_0 t+\phi)}\} = A\cos(\omega_0 t + \phi)$$

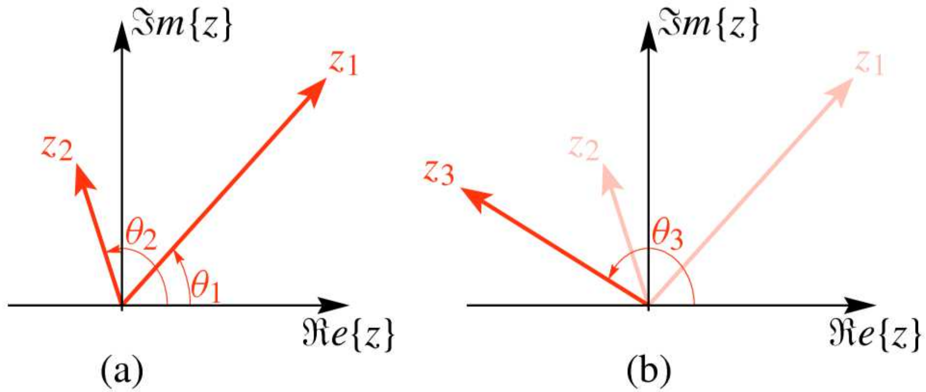
The Rotating Phasor Interpretation

- When two complex numbers are multiplied, it's best to use the polar form:

$$z_1 = r_1 e^{j\theta_1} \qquad z_2 = r_2 e^{j\theta_2}$$

$$z_3 = z_1 z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j\theta_1} e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

- We multiply the magnitudes and add the angles.
- This geometric view of complex multiplication leads to a useful interpretation of the complex exponential signal as a complex vector that rotates as time increases.



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The Rotating Phasor Interpretation

- Define the complex number

  $X = Ae^{j\phi}$

- Then $z(t) = Ae^{j(\omega_0 t + \phi)} \longrightarrow z(t) = Xe^{j\omega_0 t}$
- $z(t)$ is the product of the complex number X and the complex-valued time function $e^{j\omega_0 t}$
- X, which is called the *complex amplitude*, is a polar representation created from the *amplitude* and the *phase shift* of the complex exponential signal. The complex amplitude is also called a *phasor* (相量,相子).

The Rotating Phasor Interpretation

$$z(t) = Xe^{j\omega_0 t} = Ae^{j\phi}e^{j\omega_0 t} = Ae^{j\theta(t)}$$

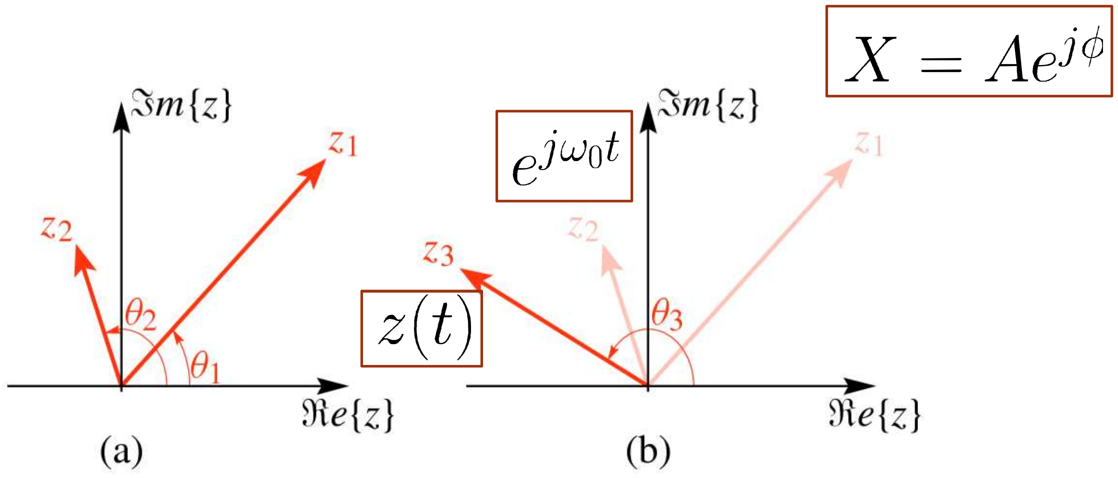
$$\theta(t) = \omega_0 t + \phi$$

- *z*(*t*) is a complex number whose magnitude is *A* and whose argument is $\theta(t)$
- *z*(*t*) can be represented as a vector in the complex plane
- The vector always lies on the perimeter of a circle of radius *A*. As *t* increases, *z*(*t*) will rotate at a constant rate, determined by ω_0

The Rotating Phasor Interpretation

$$z(t) = Xe^{j\omega_0 t} = Ae^{j\phi}e^{j\omega_0 t} = Ae^{j\theta(t)}$$

- Multiplying the phasor *X* by $e^{j\omega_0 t}$ causes the fixed phasor *X* to rotate.
  - Since $|e^{j\omega_0 t}| = 1$, no scaling occurs.
  - Another name for the complex exponential is *rotating phasor*.



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The Rotating Phasor Interpretation

- If the frequency ω_0 is positive, the direction of rotation is counterclockwise, because $\theta(t)$ will increase with increasing time.
- Rotating phasors are said to have *positive frequency* if they rotate counterclockwise, and *negative frequency* if they rotate clockwise.
- A rotating phasor makes one complete revolution every time the angle $\theta(t)$ changes by 2π radians.
- The phase shift defines where the phasor is pointing when *t*=0

$$z(t) = Xe^{j\omega_0 t} = Ae^{j\phi}e^{j\omega_0 t} = Ae^{j\theta(t)}$$

The Rotating Phasor Interpretation

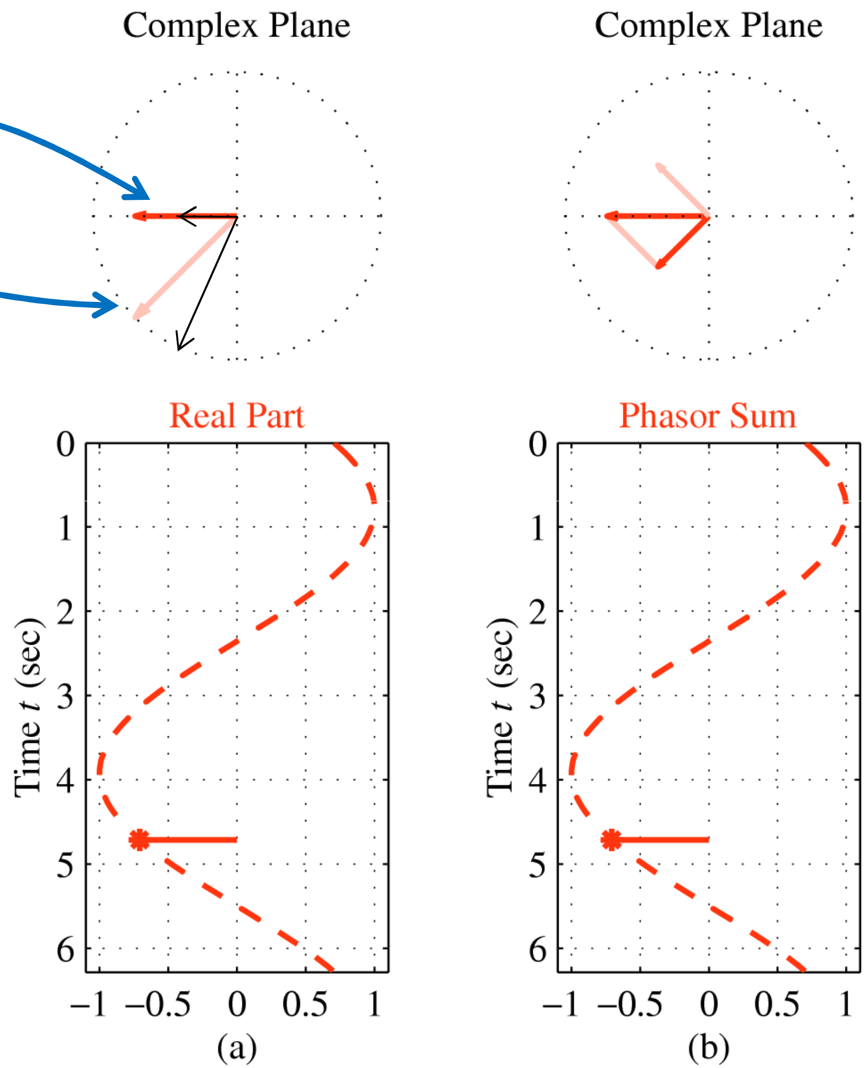
The real part of $z(t)$ at $t = 1.5\pi$

$$z(t) = e^{j(t-(\pi/4))} \text{ at } t = 1.5\pi$$

$$x(1.5\pi) = \mathfrak{Re}\{z(1.5\pi)\}$$

$$= \cos(1.5\pi - (\pi/4)) = -\frac{\sqrt{2}}{2}$$

As t increases, the rotating phasor $z(t)$ rotates in the counterclockwise direction, and its real part $x(t)$ oscillates left and right along the real axis.



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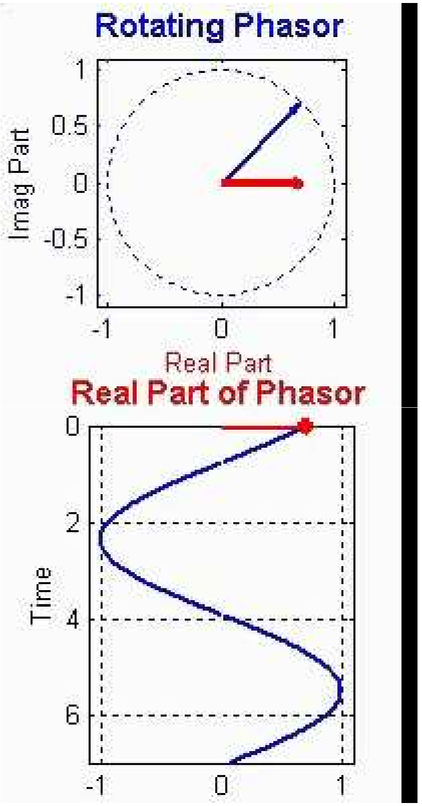
The Rotating Phasor Interpretation

- Demo



Inverse Euler Formulas

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$ (see Appendix A for details)

$$X = Ae^{j\phi}$$

$$X^* = Ae^{-j\phi}$$

$$z(t) = Xe^{j\omega_0 t}$$

$$z^*(t) = X^* e^{-j\omega_0 t}$$

$$A\cos(\omega_0 t + \phi) = A\left(\frac{e^{j(\omega_0 t + \phi)} + e^{-j(\omega_0 t + \phi)}}{2}\right)$$
$$= \tfrac{1}{2}Xe^{j\omega_0 t} + \tfrac{1}{2}X^* e^{-j\omega_0 t}$$
$$= \tfrac{1}{2}z(t) + \tfrac{1}{2}z^*(t)$$
$$= \Re\mathfrak{e}\{z(t)\}$$

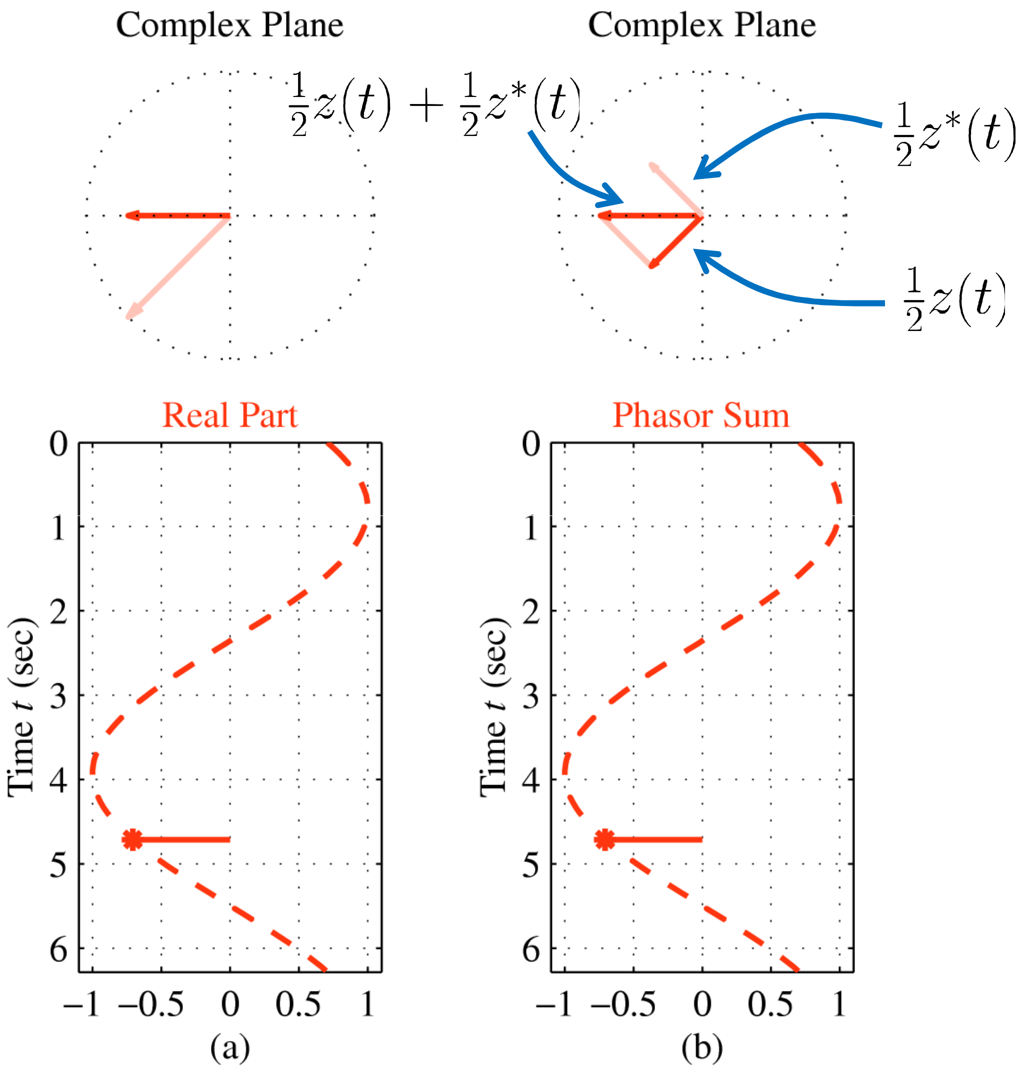
* denotes complex conjugation

The real cosine signal is actually composed of two complex exponential signals: one with positive frequency (ω_0) and the other with negative frequency ($-\omega_0$).

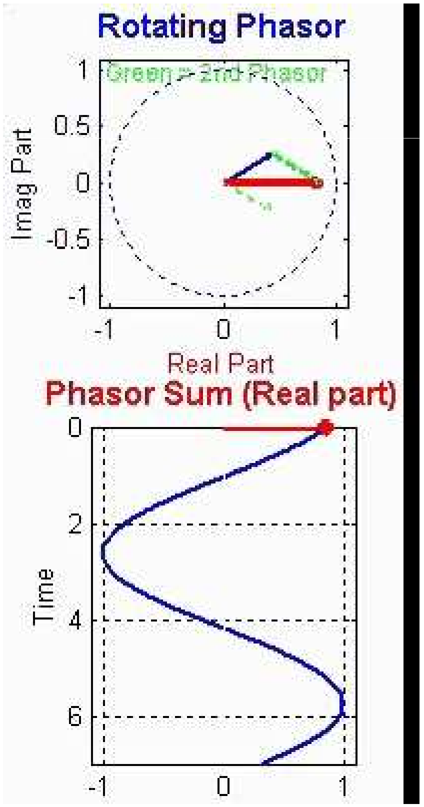
It can be represented as the sum of two complex rotating phasors that are complex conjugates of each other.

The real sine signal is also composed of two complex exponential, see Exercise 2.7

Inverse Euler Formulas



As t increases, the angle would increase in the counterclockwise direction.



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Phasor Addition

- There are many situations in which it's necessary to add two or more sinusoidal signals.
- Now we add several sinusoids having the same frequency, but with *different amplitudes and phases*.
  - A sum of *N* cosine signals of different amplitudes and phase shifts, but with the same frequency, can always be reduced to a single cosine signal of the same frequency.

$$\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = A \cos(\omega_0 t + \phi)$$

Phasor Addition

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = A\cos(\omega_0 t + \phi)$$

$$A_k \cos(\omega_0 t + \phi_k) = A_k \cos\phi_k \cos(\omega_0 t) - A_k \sin\phi_k \sin(\omega_0 t)$$

EXERCISE 2.8: Use (2.20) to show that the sum

$$1.7\cos(20\pi t + 70\pi/180) + 1.9\cos(20\pi t + 200\pi/180)$$

reduces to $A\cos(20\pi t + \phi)$, where

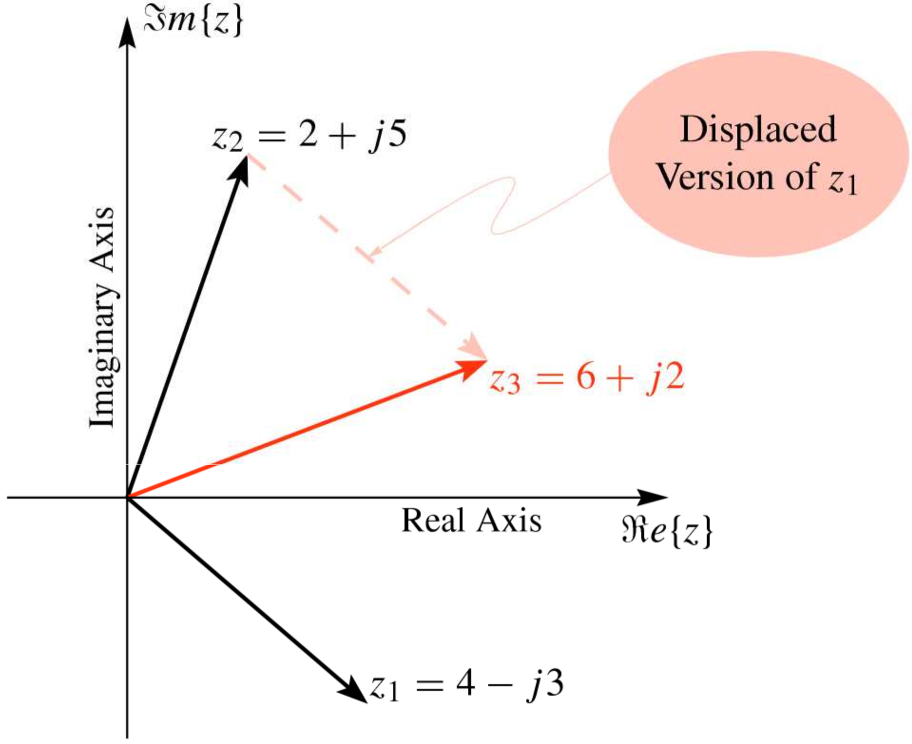
$$\begin{aligned} A &= \{[1.7\cos(70\pi/180) + 1.9\cos(200\pi/180)]^2 \\ &\quad + [1.7\sin(70\pi/180) + 1.9\sin(200\pi/180)]^2\}^{1/2} \\ &= 1.532 \end{aligned}$$

and

$$\begin{aligned} \phi &= \tan^{-1}\left\{\frac{1.7\sin(70\pi/180) + 1.9\sin(200\pi/180)}{1.7\cos(70\pi/180) + 1.9\cos(200\pi/180)}\right\} \\ &= 141.79\pi/180 = 2.475 \text{ rads.} \end{aligned}$$

The value of ϕ, given in radians, corresponds to $141.79°$.

Addition of Complex Numbers



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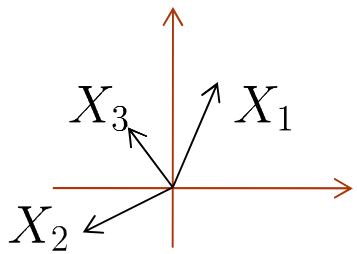
Phasor Addition Rule



$$\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = A\cos(\omega_0 t + \phi)$$

$$\boxed{\sum_{k=1}^{N} A_k e^{j\phi_k} = Ae^{j\phi}}$$

e.g. $A_1 \exp^{j\phi_1} + A_2 \exp^{j\phi_2} = X_1 + X_2 = X_3 = Ae^{j\phi}$

complex amplitude

- Any sinusoid can be written in the form:

$$A\cos(\omega_0 t + \phi) = \mathfrak{Re}\{Ae^{j(\omega_0 t + \phi)}\} = \mathfrak{Re}\{Ae^{j\phi}e^{j\omega_0 t}\}$$

- For any set of complex number $\{X_k\}$ the sum of the real parts is equal to the real part of the sum, so we have

$$\mathfrak{Re}\left\{\sum_{k=1}^{N} X_k\right\} = \sum_{k=1}^{N} \mathfrak{Re}\{X_k\}$$

Phasor Addition Rule

$$\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k)$$

$$= \sum_{k=1}^{N} \Re\mathfrak{e}\{A_k e^{j(\omega_0 t + \phi_k)}\}$$

$$= \Re\mathfrak{e}\left\{\sum_{k=1}^{N} A_k e^{j\phi_k} e^{j\omega_0 t}\right\}$$

$$= \Re\mathfrak{e}\left\{\left(\sum_{k=1}^{N} A_k e^{j\phi_k}\right) e^{j\omega_0 t}\right\}$$

$$= \Re\mathfrak{e}\{(Ae^{j\phi})e^{j\omega_0 t}\}$$

$$= \Re\mathfrak{e}\{Ae^{j(\omega_0 t + \phi)}\}$$

$$= A\cos(\omega_0 t + \phi)$$

$$\sum_{k=1}^{N} A_k e^{j\phi_k} = Ae^{j\phi}$$

Example

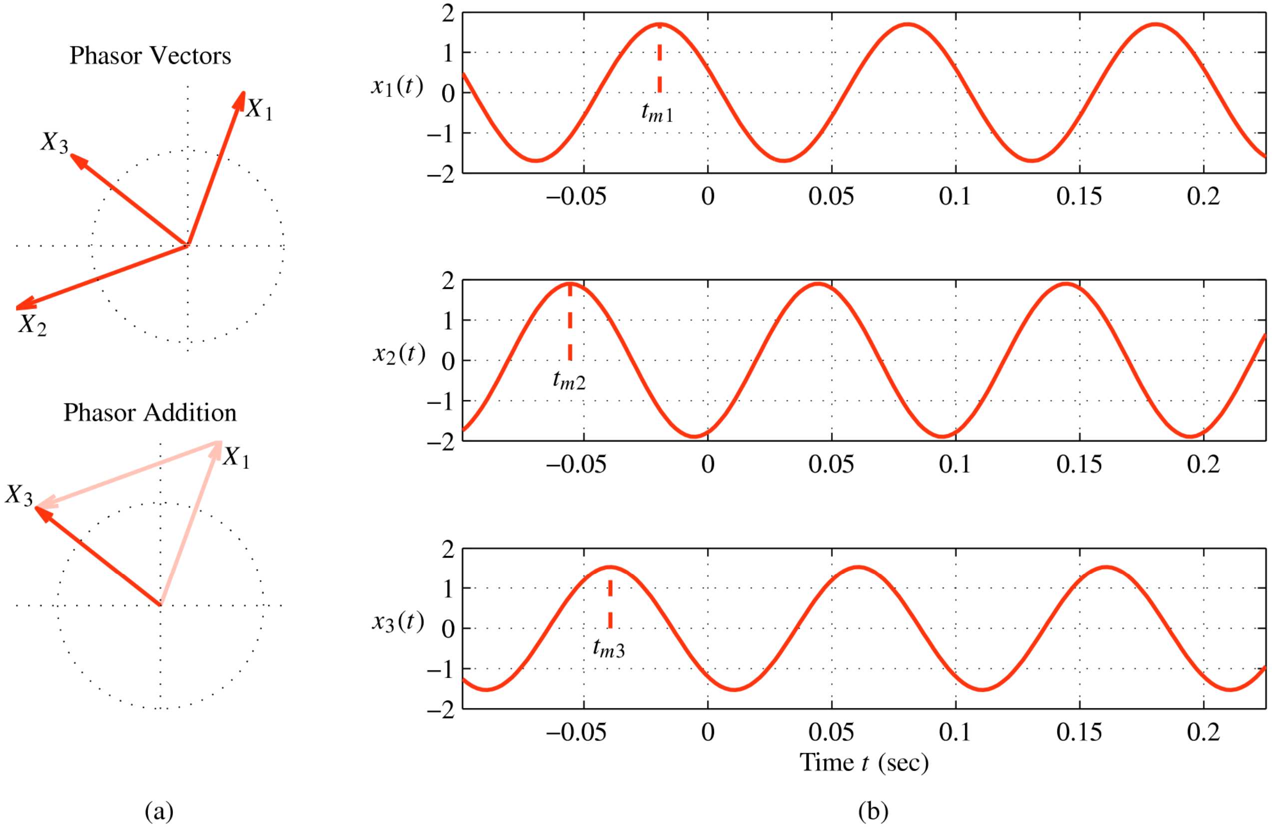
$$x_1(t) = 1.7\cos(20\pi t + 70\pi/180) \qquad x_2(t) = 1.9\cos(20\pi t + 200\pi/180)$$

$$x_3(t) = x_1(t) + x_2(t) = 1.532\cos(20\pi t + 141.79\pi/180)$$

The frequency is 10 Hz, so the period is T_0=0.1 sec.

$$t_m = -\frac{\phi T_0}{2\pi}$$

t_{m1} = -0.0194
t_{m2} = -0.0556
t_{m3} = -0.0394



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Example

$$x_1(t) = 1.7\cos(20\pi t + 70\pi/180)$$
$$x_2(t) = 1.9\cos(20\pi t + 200\pi/180)$$

- (a) Represent $x_1(t)$ and $x_2(t)$ by the phasors:

$$X_1 = A_1 e^{j\phi_1} = 1.7e^{j70\pi/180} \qquad X_2 = A_2 e^{j\phi_2} = 1.9e^{j200\pi/180}$$

- (b) Convert both phasors to rectangular form:

$$X_1 = 0.5814 + j1.597 \qquad X_2 = -1.785 - j0.6498$$

- (c) Add the real parts and the imaginary parts:

$$X_3 = X_1 + X_2 = (0.5814 + j1.597) + (-1.785 - j0.6498)$$
$$= -1.204 + j0.9476$$

- (d) Convert back to polar form, obtaining

$$X_3 = 1.532e^{j141.79\pi/180}$$

- The final formula for $x_3(t)$ is

$$x_3(t) = 1.532\cos(20\pi t + 141.79\pi/180)$$

Summary of the Phasor Addition Rule

$\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = A\cos(\omega_0 t + \phi)$

- (a) Obtain the phasor representation $X_k = A_k e^{j\phi_k}$ of each of the individual signals
- (b) Add the phasors of the individual signals to get $X = X_1 + X_2 + \cdots = Ae^{j\phi}$. This requires polar-to-Cartesian-to polar format conversions.
- (c) Multiply X by $e^{j\omega_0 t}$ to get $z(t) = Ae^{j\phi}e^{j\omega_0 t}$
- (d) Take the real part to get

$$x(t) = \Re\mathfrak{e}\{Ae^{j\phi}e^{j\omega_0 t}\}$$
$$= A\cos(\omega_0 t + \phi) = x_1(t) + x_2(t) + \cdots$$

$$\begin{aligned}
&\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) \\
&= \sum_{k=1}^{N} \Re\mathfrak{e}\{A_k e^{j(\omega_0 t + \phi_k)}\} \\
&= \Re\mathfrak{e}\left\{\sum_{k=1}^{N} A_k e^{j\phi_k} e^{j\omega_0 t}\right\} \\
&= \Re\mathfrak{e}\left\{\left(\sum_{k=1}^{N} A_k e^{j\phi_k}\right) e^{j\omega_0 t}\right\} \\
&= \Re\mathfrak{e}\{(Ae^{j\phi})e^{j\omega_0 t}\} \\
&= \Re\mathfrak{e}\{Ae^{j(\omega_0 t + \phi)}\} \\
&= A\cos(\omega_0 t + \phi)
\end{aligned}$$

Homework 1

- Exercises 2.7, 2.9, 2.14, 2.16
- Hand over your homework at the class of Mar. 8

Lecture 3 Spectrum Representation

Fundamentals of Digital Signal Processing
Spring, 2012

Wei-Ta Chu
2012/3/1

Spectrum (頻譜)

- Spectrum: a compact representation of the frequency content of a signal that is composed of sinusoids.
- We will show how more complicated waveforms can be constructed out of sums of sinusoidal signals of *different amplitudes, phases, and frequencies*.
- A signal is created by adding a constant and *N* sinusoids $x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$
- Such a signal may be represented in terms of the complex amplitude $x(t) = X_0 + \sum_{k=1}^{N} \Re e\{X_k e^{j2\pi f_k t}\}$

$$X_k = A_k e^{j\phi_k}$$

$$X_0 = A_0$$

Spectrum

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\{X_k e^{j2\pi f_k t}\}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

- By using inverse Euler formula

$$X_k = A_k e^{j\phi_k}$$

$$X_k^* = A_k e^{-j\phi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t} \right\}$$

- The real part of a complex number is equal to one-half the sum of that number and its complex conjugate.
- Each sinusoid in the sum decomposes into two rotating phasors, one with positive frequency, f_k , and the other with negative frequency, $-f_k$.

Spectrum

- Two-sided spectrum of a signal composed of sinusoids as
  $x(t) = X_0 + \sum_{k=1}^{N}\left\{\frac{X_k}{2}e^{j2\pi f_k t} + \frac{X_k^*}{2}e^{-j2\pi f_k t}\right\}$

- Set of pairs
  $\{(0, X_0), (f_1, \frac{1}{2}X_1), (-f_1, \frac{1}{2}X_1^*), \cdots, (f_k, \frac{1}{2}X_k), (-f_k, \frac{1}{2}X_k^*)\}$
- Each pair indicates the size and relative phase of the sinusoidal component contributing at frequency f_k
- *Frequency-domain representation* of the signal

Example

$$14\cos(200\pi t - \pi/3) = 14 * \left(\frac{e^{j(200\pi t - \pi/3)} + e^{-j(200\pi t - \pi/3)}}{2}\right)$$
$$= 7e^{-j\pi/3}e^{j200\pi t} + 7e^{j\pi/3}e^{-j200\pi t}$$

$$x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$$

- Apply inverse Euler formula

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$
$$+ 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

- The constant component of the signal, often called the *DC component*, can be expressed as $10e^{j0t} = 10$.
- The spectrum of the signal is represented by

$$\{(0, 10), (100, 7e^{-j\pi/3}), (-100, 7e^{j\pi/3}), (250, 4e^{j\pi/2}), (-250, 4e^{-j\pi/2})\}$$

Notation Change

- Introduce a new symbol for the complex amplitude in the spectrum

$$a_k = \begin{cases} A_0 & \text{for } k = 0 \\ \frac{1}{2}A_k e^{j\phi_k} = \frac{1}{2}X_k & \text{for } k \neq 0 \end{cases}$$

- The spectrum is the set of (f_k, a_k) pairs
- Define $f_0 = 0$. Now we can rewrite

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t} \right\}$$

➡ $$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

$$X_k = A_k e^{j\phi_k}$$

$$a_{-k} = a_k^*$$

Graphical Plot of the Spectrum

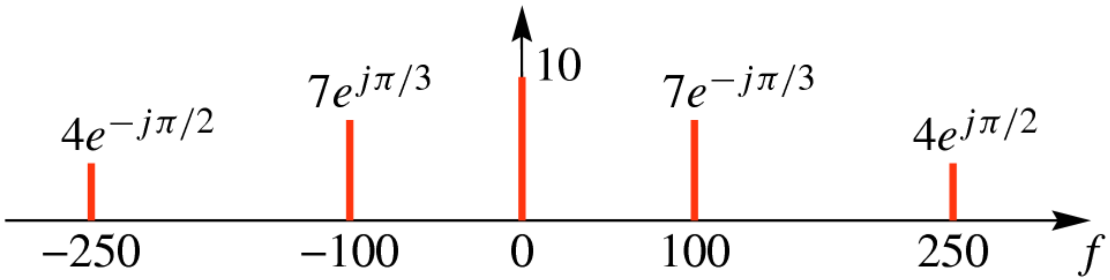
- Spectrum plot of

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

$$x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$$

$$\{(0,10), (100, 7e^{-j\pi/3}), (-100, 7e^{j\pi/3}), (250, 4e^{j\pi/2}), (-250, 4e^{-j\pi/2})\}$$

- This plot shows the relative location of the frequencies, and the relative amplitudes of the sinusoidal components
- The rotating phasors with positive and negative frequency must combine to form a real signal



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Spectrum

- A general procedure for computing and plotting the spectrum for an arbitrarily chosen signal requires the study of *Fourier analysis*.
- If it's known a priori that a signal is composed of a finite number of sinusoidal components, the process of analyzing that signal to find its *spectral components* involves writing an equation for the signal in the form of

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t} \right\}$$

and picking off the amplitude, phase, and frequency of each of its rotating phasor components