Lecture 20 IIR Filters 2

Fundamentals of Digital Signal Processing

Spring, 2012

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2012/5/15

Relation to The Impulse Response

- In the IIR case, we need to take the *z*-transform of an infinitely long sequence to obtain the system function.
- Consider $h[n] = a^n u[n]$
  $H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$
- If $|az^{-1}| < 1$, then the sum is finite, and in fact is given by the closed-form expression
  $H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \dfrac{1}{1 - az^{-1}}$
- The condition for the infinite sum to equal the closed-form expression can be expressed as |*a*|< *z*. This condition is called the *region of convergence*.

$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

p. 165, property (7.15)

Relation to The Impulse Response

$$\boxed{a^n u[n] \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1-az^{-1}}}$$

- Example:

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$h[n] = b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]$$

- Using the linearity property and delay property of the z-transform

$$H(z) = b_0 \frac{1}{1-a_1 z^{-1}} + b_1 z^{-1} \frac{1}{1-a_1 z^{-1}}$$

$$= \frac{b_0 + b_1 z^{-1}}{1-a_1 z^{-1}}$$

The same as the one we obtained in p. 205

Summary of The Method

- It's possible to go from the difference equation directly to the system function
- It's possible by taking the inverse z-transform to go directly from the system function to the impulse response.
- It's possible to do this by a process of inverse z-transformation based on breaking the z-transform into a sum of terms like $\frac{1}{1 - az^{-1}}$

Poles and Zeros

- For a system function, the zero locations in the complex *z*-plane are very important for characterizing the system.
- It's probably better for finding roots to rewrite the polynomials as functions of *z* rather than z^{-1}.

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 z + b_1}{z - a_1}$$

$$b_0 z + b_1 = 0 \Rightarrow \text{Root at } z = -\frac{b_1}{b_0}$$

$$z - a_1 = 0 \Rightarrow \text{Root at } z = a_1$$

Zero of the function H(z)

$$H(z)\big|_{z=-(b_1/b_0)} = 0$$

Pole of the function H(z)

$$H(z)\big|_{z=a_1} \rightarrow \infty$$

Exercise

EXERCISE 8.9: Find the poles and zeros of the following z-transform system function

$$H(z) = \frac{3 + 4z^{-1}}{1 + 0.5z^{-1}}$$

$$H(z) = \frac{3 + 4z^{-1}}{1 + \frac{1}{2}z^{-1}}$$

Multiply numerator and denominator by z

$$H(z) = \frac{3z + 4}{z + 1/2}$$

Numerator root: $3z + 4 = 0$

$$3z = -4$$

$$\boxed{z = -4/3 \quad (\text{zero})}$$

Denominator root: $z + \frac{1}{2} = 0$

$$\boxed{z = -\frac{1}{2} \quad (\text{pole})}$$

Poles or Zeros at The Origin or Infinity

- *The number of poles equals the number of zeros*.
- Consider $y[n] = 0.5y[n-1] + 2x[n]$

$$H(z) = \frac{2}{1-0.5z^{-1}} \Rightarrow H(z) = \frac{2z}{z-0.5}$$

We see that there is one pole at *z*=0.5 and a zero at *z*=0.

- Consider $y[n] = 0.5y[n-1] + 3x[n-1]$

$$H(z) = \frac{3z^{-1}}{1-0.5z^{-1}} = \frac{3}{z-0.5}$$

The system has one pole at z = 0.5.

If we take the limit of *H*(z) as $z \to \infty$, we get $H(z) \to 0$

Thus, it also has one zero at $z = \infty$

Pole Locations and Stability

- The pole location of a first-order filter determines the shape of the impulse response.

$$H(z) = b_0\left(\frac{1}{1-a_1z^{-1}}\right) + b_1z^{-1}\left(\frac{1}{1-a_1z^{-1}}\right)$$

$$= \frac{b_0 + b_1z^{-1}}{1-a_1z^{-1}} = \frac{b_0(z+b_1/b_0)}{z-a_1}$$

has an impulse response

$$h[n] = b_0(a_1)^n u[n] + b_1(a_1)^{n-1}u[n-1]$$

$$= \begin{cases} 0 & \text{for } n < 0 \\ b_0 & \text{for } n = 0 \\ (b_0 + b_1a_1^{-1})a_1^n & \text{for } n \geq 1 \end{cases}$$

Pole Locations and Stability

- An IIR system with a single pole at $z=a_1$ has an impulse response that is proportional to a_1^n for $n > 1$.
- We see that if $|a_1|<1$, the impulse response will die out as $n \to \infty$.
- If $|a_1| \geq 1$, the impulse response will not die out; in fact if $|a_1|>1$, it will grow without bound.
- Since the pole of the system function is at $z=a_1$, we see that the location of the pole can tell us whether the impulse response will decay or grow.

Pole Locations and Stability

- Systems that produce bounded outputs when the input is bounded are called *stable systems*. If $|a_1|<1$, the pole of the system function is *inside* the unit circle of the z-plane.

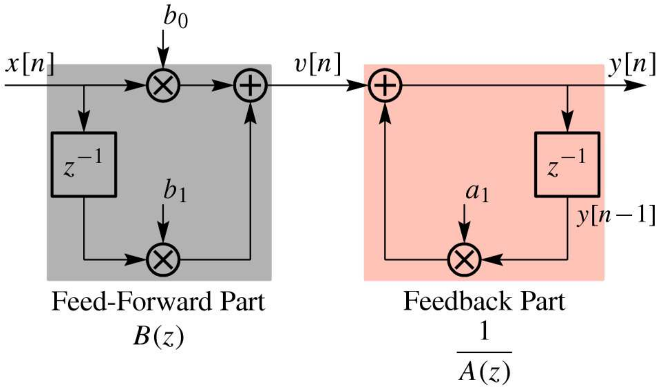
A causal LTI IIR system with initial rest conditions is stable if all of the poles of its system function lie strictly inside the unit circle of the z-plane.

- Thus, stability of a system can be seen at a glance from a z-plane plot of the poles and zeros of the system function.

Example

$$H(z)=\frac{1-2z^{-1}}{1-0.8z^{-1}}=\frac{z-2}{z-0.8}$$

- Has a zero at $z = 2$ and a pole at $z = 0.8$. Therefore, the system is stable.
- Recall that the zeros correspond to an FIR system that is in cascade with an IIR system defined by the poles. Since FIR systems are always stable, it is not surprising that stability is determined solely by the poles of the system function. (p. 206)



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Frequency Response of an IIR Filter

- Recall the frequency response of FIR systems
- If the input $x[n] = e^{j\hat{\omega}n}$, then $y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$
- Relate to the system function

  $H(e^{j\hat{\omega}}) = H(z)|_{z=e^{j\hat{\omega}}}$

- The relation between the system function and the frequency response also holds for IIR systems. However, we must add the provision that the system must be stable in order for the frequency response to exist.

Frequency Response of an IIR Filter

- Recall that the system function for the general first-order IIR system has the form

  $$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

  where the region of convergence of the system function is $|a_1 z^{-1}| < 1$ or $|a_1| < |z|$.
- If we wish to evaluate *H*(*z*) for $z = e^{j\hat{\omega}}$, then the values of *z* on the unit circle should be in the region of convergence; i.e., we require |z|=1 to be in the region of convergence of the *z*-transform.

Frequency Response of an IIR Filter

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$

- It's a periodic function with a period equal to 2π
- We can reduce this equation to two separate real formulas for the magnitude and the phase as functions of frequency.
- For magnitude response, it's expedient to compute the magnitude squared first, and then take a square root if necessary.

$$\begin{aligned}
|H(e^{j\hat{\omega}})|^2 &= H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}}) \\
&= \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}} \cdot \frac{b_0^* + b_1^* e^{+j\hat{\omega}}}{1 - a_1^* e^{+j\hat{\omega}}} \\
&= \frac{|b_0|^2 + |b_1|^2 + b_0 b_1^* e^{+j\hat{\omega}} + b_0^* b_1 e^{-j\hat{\omega}}}{1 + |a_1|^2 - a_1^* e^{+j\hat{\omega}} - a_1 e^{-j\hat{\omega}}} \\
&= \frac{|b_0|^2 + |b_1|^2 + 2\mathfrak{Re}\{b_0^* b_1 e^{-j\hat{\omega}}\}}{1 + |a_1|^2 - 2\mathfrak{Re}\{a_1 e^{-j\hat{\omega}}\}}
\end{aligned}$$

Frequency Response of an IIR Filter

- This derivation does not assume that the filter coefficients are real. If the coefficients were real, we would get the further simplification

$$|H(e^{j\hat{\omega}})|^2 = \frac{|b_0|^2 + |b_1|^2 + 2b_0b_1\cos(\hat{\omega})}{1 + |a_1|^2 - 2a_1\cos(\hat{\omega})}$$

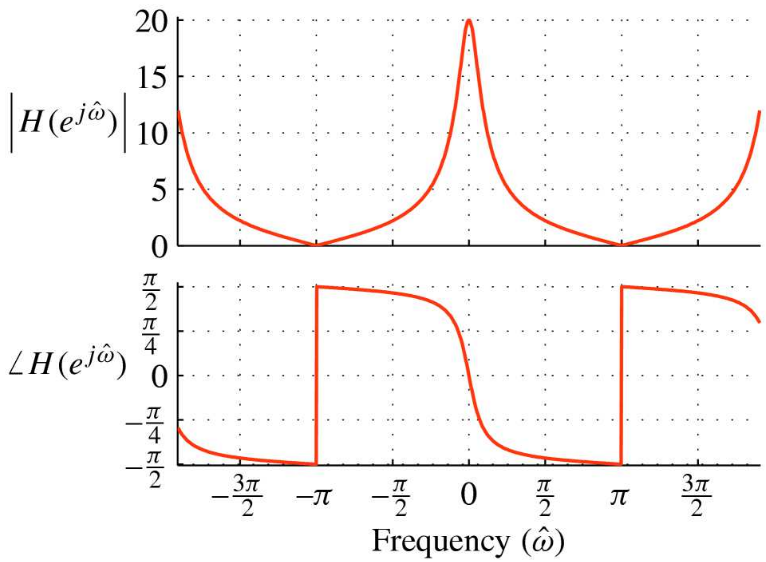
- The phase response is even messier. Arctangents would be used to extract the angles of the numerator and denominator, and then the two phases would be subtracted.

$$\phi(\hat{\omega}) = \tan^{-1}\Big(\frac{-b_1\sin\hat{\omega}}{b_0 + b_1\cos\hat{\omega}}\Big) - \tan^{-1}\Big(\frac{a_1\sin\hat{\omega}}{1 - a_1\cos\hat{\omega}}\Big)$$

Frequency Response Using Matlab

- In Matlab, the frequency response is evaluated over an equally spaced gird in the $\hat{\omega}$ domain, and then its magnitude and phase can be plotted.
- Consider $y[n] = 0.8y[n-1] + 2x[n] + 2x[n-1]$

$$\Rightarrow \quad y[n] - 0.8y[n-1] = 2x[n] + 2x[n-1]$$



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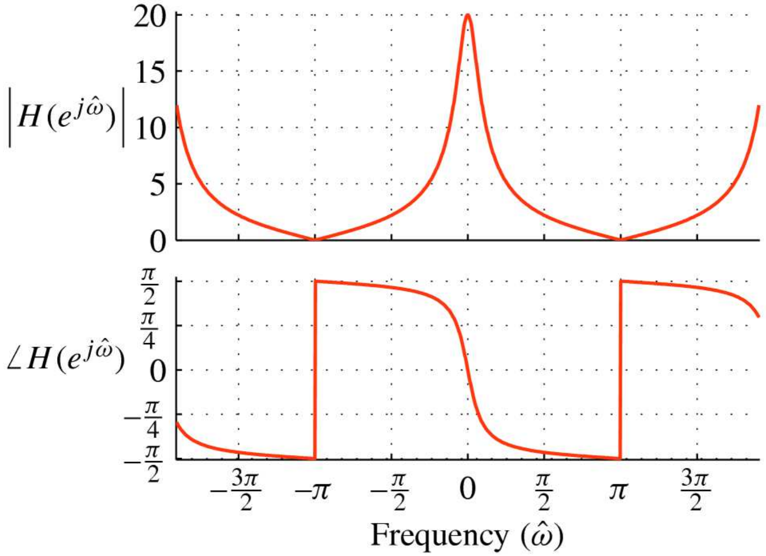
Frequency Response Using Matlab

- For this system, the system function is $$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

- A pole at z = 0.8, and a zero at z = -1

$$z = -1 = e^{j\pi} = e^{j\hat{\omega}}\Big|_{\hat{\omega}=\pi}$$

- Thus, $H(e^{j\hat{\omega}})$ has the value zero at $\hat{\omega} = \pi$, since $H(z)$ is zero at z = -1.



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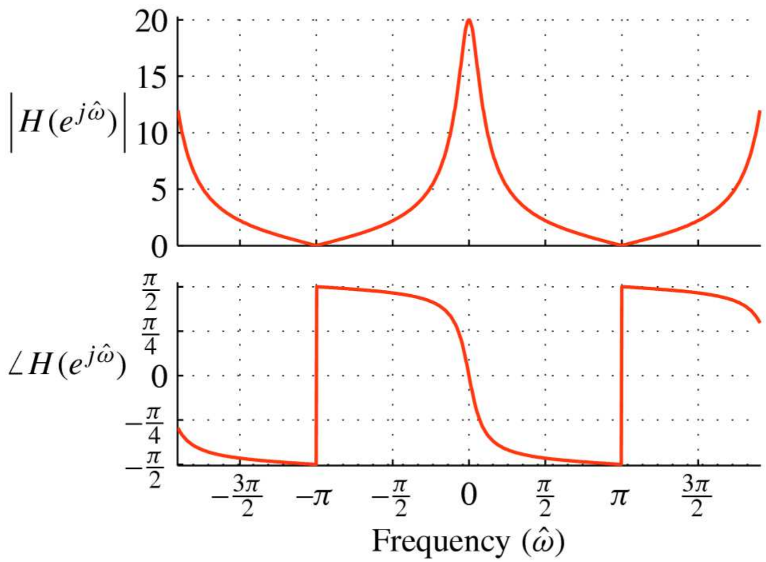
Frequency Response Using Matlab

- $H(z)$ blows up at $z = 0.8$, the nearby points on the unit circle must have large values.
- The closest point on the unit circle is at $z = e^{j0} = 1$. In this case, we can evaluate the frequency response directly from the formula to get

$$H(e^{j\hat{\omega}})\Big|_{\hat{\omega}=0} = H(z)\Big|_{z=1}$$

$$= \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}\Big|_{z=1} = 20$$

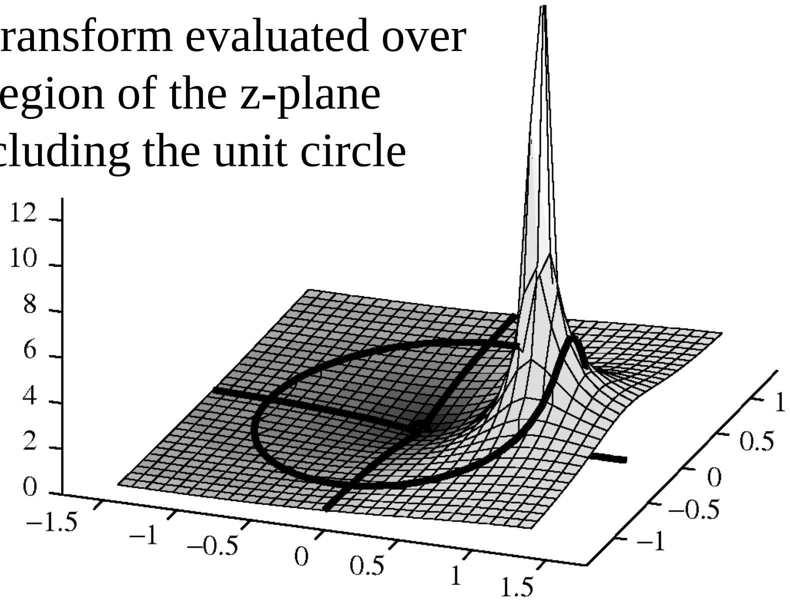


Three-Dimensional Plot

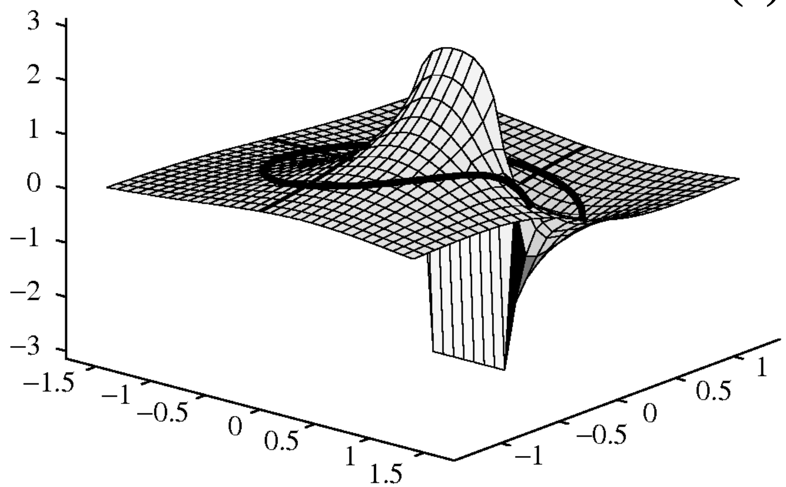
- Consider the system function $H(z) = \dfrac{1}{1 - 0.8z^{-1}}$ Zero at $z = 0$ Pole at $z = 0.8$

- The pole (at z = 0.8) creates a large peak that makes all nearby values very large.
- Most rapid transition at $\hat{\omega} = 0$, which is $z = 1$ (the closest point on the unit circle to the pole at z = 0.8)

z-transform evaluated over a region of the z-plane including the unit circle

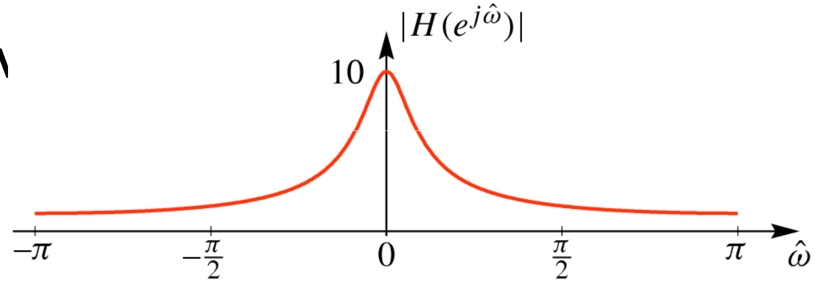


Phase of H(z)

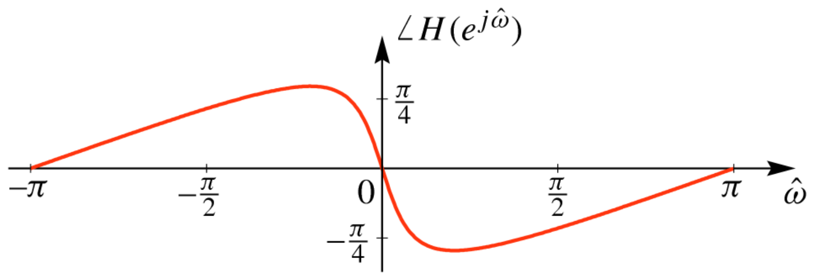


Three-Dimensional Plot

- The frequency response $H(e^{j\hat{\omega}})$ is obtained by selecting the values of $H(z)$ along the unit circle.
- The pole at z=0.8 pushes $H(e^{j\hat{\omega}})$ up in the region near $\hat{\omega} = 0$, which is the same as z = 1.
- The unit circle values follow the ups and downs of $H(z)$ as $\hat{\omega}$ goes from $-\pi$ to $+\pi$



(a)



(b)

Three Domains

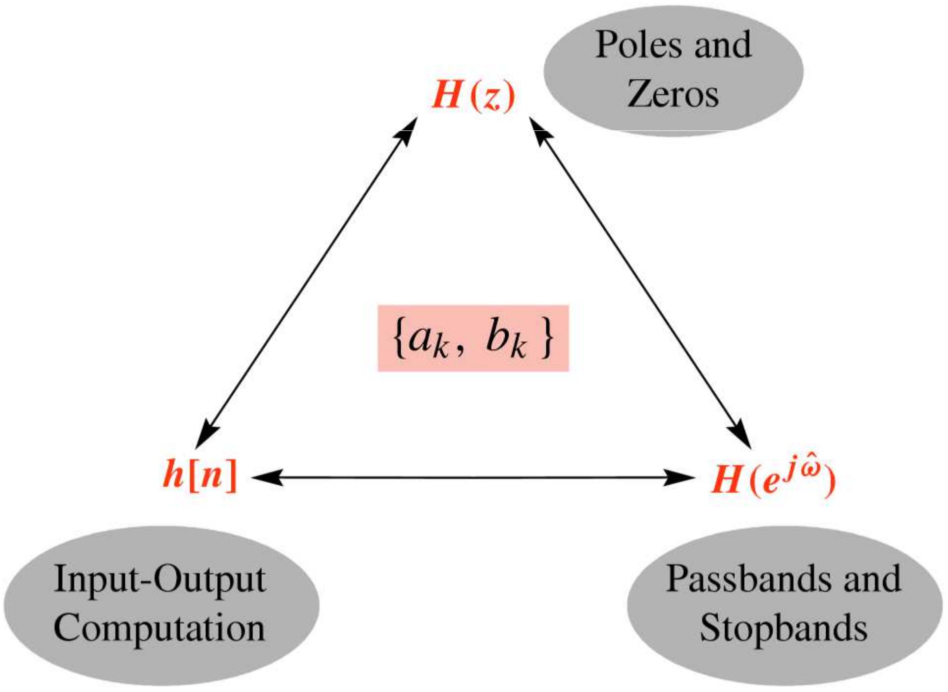
- For the second-order case

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

- This equation provides the algorithm for computing the output signal from the input signal by iteration using the filter coefficients $\{a_1, a_2, b_0, b_1, b_2\}$. It also defines the impulse response $h[n]$.
- System function $H(z) = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$
- Frequency response $H(z) = \dfrac{b_0 + b_1 e^{-j\hat{\omega}} + b_2 e^{-j2\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}} - a_2 e^{-j2\hat{\omega}}}$

Three Domains

- The shapes of the passbands and stopbands of the frequency response are highly dependent on the pole and zero locations.
- The character of the impulse response can be related to the poles.
  - Getting $h[n]$ directly from $H(z)$



The Inverse z-Transform

- Consider a system function

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

- Recall that the *z*-transform of the output of this system is $Y(z) = H(z)X(z)$, (Sec. 8-2.4) so one approach to finding the output for a given input $x[n]$ is as follows:
  - 1. Determine the *z*-transform $X(z)$ of the input signal $x[n]$
  - 2. Multiply $X(z)$ by $H(z)$ to get $Y(z)$
  - 3. Determine the inverse *z*-transform of $Y(z)$ to get the output $y[n]$

The Inverse z-Transform

- We will derive a general procedure for the inverse *z*-transform
- In the case of the step response, we see that the input $x[n] = u[n]$. From (8.27), the *z*-transform of $x[n] = u[n]$ is $X(z) = \dfrac{1}{1 - z^{-1}}$
- So $$Y(z) = H(z)X(z) = \frac{b_0 + b_1 z^{-1}}{(1 - a_1 z^{-1})(1 - z^{-1})}$$ $$= \frac{b_0 + b_1 z^{-1}}{1 - (1 + a_1)z^{-1} + a_1 z^{-2}}$$

The Inverse z-Transform

$$Y(z) = H(z)X(z) = \frac{b_0 + b_1 z^{-1}}{(1 - a_1 z^{-1})(1 - z^{-1})}$$

$$= \frac{b_0 + b_1 z^{-1}}{1 - (1 + a_1)z^{-1} + a_1 z^{-2}}$$

- Go back to the *n*-domain by inverse transformation
  - A standard approach is to use a table of z-transform pairs and simply look up the answer in the table

Table 8-1: Summary of important z-transform properties and pairs.

	SHORT TABLE OF z-TRANSFORMS		
	$x[n]$	$\overset{z}{\longleftrightarrow}$	$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\overset{z}{\longleftrightarrow}$	$aX_1(z) + bX_2(z)$
2.	$x[n - n_0]$	$\overset{z}{\longleftrightarrow}$	$z^{-n_0}X(z)$
3.	$y[n] = x[n] * h[n]$	$\overset{z}{\longleftrightarrow}$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\overset{z}{\longleftrightarrow}$	1
5.	$\delta[n - n_0]$	$\overset{z}{\longleftrightarrow}$	z^{-n_0}
6.	$a^n u[n]$	$\overset{z}{\longleftrightarrow}$	$\frac{1}{1 - az^{-1}}$

Partial Fraction Expansion

- Use *partial fraction expansion* (部份分式法) of $Y(z)$
- The technique is based on the observation that a rational function $Y(z)$ can be expressed as a sum of simpler rational functions.

$$Y(z) = H(z)X(z) = \frac{b_0 + b_1 z^{-1}}{(1 - a_1 z^{-1})(1 - z^{-1})}$$

$$= \frac{A}{1 - a_1 z^{-1}} + \frac{B}{1 - z^{-1}}$$

Partial Fraction Expansion

$$Y(z) = H(z)X(z) = \frac{b_0 + b_1 z^{-1}}{(1 - a_1 z^{-1})(1 - z^{-1})}$$

$$= \frac{A}{1 - a_1 z^{-1}} + \frac{B}{1 - z^{-1}}$$

$$\Rightarrow \quad Y(z)(1 - a_1 z^{-1}) = \frac{b_0 + b_1 z^{-1}}{(1 - z^{-1})} = A + \frac{B(1 - a_1 z^{-1})}{1 - z^{-1}}$$

$$\Rightarrow \quad Y(z)(1 - a_1 z^{-1})\Big|_{z=a_1} = \frac{b_0 + b_1 z^{-1}}{(1 - z^{-1})}\Big|_{z=a_1} = A + \frac{B(1 - a_1 z^{-1})}{1 - z^{-1}}\Big|_{z=a_1} = A$$

$$\Rightarrow \quad A = Y(z)(1 - a_1 z^{-1})\Big|_{z=a_1} = \frac{b_0 + b_1 z^{-1}}{(1 - z^{-1})}\Big|_{z=a_1} = \frac{b_0 + b_1 a_1^{-1}}{(1 - a_1^{-1})}$$

$$\Rightarrow \quad B = Y(z)(1 - z^{-1})\Big|_{z=1} = \frac{b_0 + b_1}{1 - a_1}$$

Partial Fraction Expansion

- Using the superposition property (entry 1 in Table 8-1), and the exponential z-transform pair (entry 6 in Table 8-1), we can write down the desired answer as

$$y[n] = \left(\frac{b_0 + b_1 a_1^{-1}}{1 - a_1^{-1}}\right) a_1^n u[n] + \left(\frac{b_0 + b_1}{1 - a_1}\right) u[n]$$

$$y[n] = \left(\frac{(b_0 + b_1) - (b_0 a_1 + b_1) a_1^n}{1 - a_1}\right) u[n]$$

- If we substitute the value $b_1 = 0$, we get

$$y[n] = b_0 \left(\frac{1 - a_1^{n+1}}{1 - a_1}\right) u[n]$$

which is the same as (8.16) and (8.18) in Sec. 8-2.4

A General Procedure for Inverse z-Transformation

- Let $X(z)$ be any rational z-transform of degree N in the denominator and M in the numerator. Assuming that $M < N$, we can find the sequence $x[n]$ that corresponds to $X(z)$ by $(1 - p_k z^{-1})$
- 1. Factor the denominator polynomial of $H(z)$ and express the pole factors in the form for $k = 1, 2, \ldots, N$
- 2. Make a partial fraction expansion of $H(z)$ into a sum of terms of the form

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \qquad A_k = H(z)(1 - p_k z^{-1})\Big|_{z=p_k}$$

- 3. Write down the answer as

$$h[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n]$$