Lecture 13 Frequency Response of FIR Filters

Fundamentals of Digital Signal Processing
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Frequency Response

- If we can find a representation for a signal in terms of complex exponentials, the frequency response gives a simple and highly intuitive means for determining what an LTI system does to that input signal.
- If the input is $x[n] = X_0 + \sum_{k=1}^{N} \left(\frac{X_k}{2} e^{j\hat{\omega}_k n} + \frac{X_k^*}{2} e^{-j\hat{\omega}_k n} \right)$
  $= X_0 + \sum_{k=1}^{N} |X_k| \cos(\hat{\omega}_k n + \angle X_k)$
- The corresponding output is

$$y[n] = H(e^{j0})X_0 + \sum_{k=1}^{N} \left(H(e^{j\hat{\omega}_k}) \frac{X_k}{2} e^{j\hat{\omega}_k n} + H(e^{-j\hat{\omega}_k}) \frac{X_k^*}{2} e^{-j\hat{\omega}_k n} \right)$$
$$= H(e^{j0})X_0 + \sum_{k=1}^{N} |H(e^{j\hat{\omega}_k})||X_k| \cos\Big(\hat{\omega}_k n + \angle X_k + \angle H(e^{j\hat{\omega}_k}) \Big)$$

Each individual complex exponential component is modified by the frequency response evaluated at the frequency of that component.

Example

- For the FIR filter with coefficients $\{b_k\}=\{1,2,1\}$, find the output when the input is

$$x[n] = 4 + 3\cos\left(\frac{\pi}{3}n - \frac{\pi}{2}\right) + 3\cos\left(\frac{7\pi}{8}n\right)$$

- We evaluate $H(e^{j\hat{\omega}})$ at frequencies 0, $\pi/3$, and $7\pi/8$

$$H(e^{j0}) = 4$$
$$H(e^{j\pi/3}) = 3e^{-j\pi/3}$$
$$H(e^{j7\pi/8}) = 0.1522e^{-j7\pi/8}$$

$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

- Output

$$y[n] = 4\cdot 4 + 3\cdot 3\cos\left(\frac{\pi}{3}n - \frac{\pi}{3} - \frac{\pi}{2}\right) + 0.1522\cdot 3\cos\left(\frac{7\pi}{8}n - \frac{7\pi}{8}\right)$$
$$= 16 + 9\cos\left(\frac{\pi}{3}(n-1) - \frac{\pi}{2}\right) + 0.4567\cos\left(\frac{7\pi}{8}(n-1)\right)$$

Frequency Domain vs. Time Domain

- The examples described above are called the *frequency-domain* approach.
  - We do not need to deal with the *time-domain* description (i.e., the difference equation or impulse response) of the system when the input is complex exponential signal.
- We think about how the spectrum of the signal is modified by the system rather than considering what happens to the individual samples of the input signal.

Steady-State and Transient Response

- If the input is

$$x[n] = Xe^{j\hat{\omega}n} \qquad X = Ae^{j\phi} \qquad -\infty < n < \infty$$

- Then the corresponding output of an LTI FIR system is

$$y[n] = H(e^{j\hat{\omega}})Xe^{j\hat{\omega}n} \qquad -\infty < n < \infty$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

- The condition that $x[n]$ be a complex exponential signal existing over $-\infty < n < \infty$ is important. Without this condition, we will not obtain the simple result of $y[n]$.
- However, this condition appears to be somewhat impractical.

Steady-State and Transient Response

- Consider the following "suddenly applied" complex exponential signal that starts at n=0 and is nonzero only for $n \geq 0$

$$x[n] = Xe^{j\hat{\omega}n}u[n] = \begin{cases} Xe^{j\hat{\omega}n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

- The output of an LTI FIR system for this input is

$$y[n] = \sum_{k=0}^{M} b_k X e^{j\hat{\omega}(n-k)} u[n-k]$$

Steady-State and Transient Response

- By considering different values of *n* and the fact that *u*[*n*-*k*] = 0 for *k*>*n*, it follows that the sum can be expressed as

$$y[n] = \begin{cases} 0 & n < 0 \\ \left(\sum_{k=0}^{n} b_k e^{-j\hat{\omega}k}\right) X e^{j\hat{\omega}n} & 0 \le n < M \\ \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right) X e^{j\hat{\omega}n} & M \le n \end{cases}$$

- When the complex exponential signal is suddenly applied, the output can be considered to be defined over three distinct regions.

Steady-State and Transient Response

- (1) For *n*<0, the input is zero, and therefore the corresponding output is zero.
- (2) The second region is a transition region whose length is *M* samples. The complex multiplier of $e^{j\hat{\omega}n}$ depends upon *n*. This region is called the *transient* part of the output.
- (3) In the third region, $M < n$, the output is identical to the output that would be obtained if the input were defined over the doubly infinite interval.

$$y[n] = H(e^{j\hat{\omega}})Xe^{j\hat{\omega}n} \qquad M < n$$

It's called the *steady-state* part.

Example

- Consider the filter coefficients are $\{b_k\} = \{1,-2.4,-2,1\}$
- The frequency response of this system is

$$H(e^{j\hat{\omega}}) = [4 - 4\cos(\hat{\omega}) + 2\cos(2\hat{\omega})]e^{-j2\hat{\omega}}$$

- If the input is the suddenly applied cosine signal

$$x[n] = \cos(0.2\pi n - \pi)u[n]$$

$$H(e^{j\hat{\omega}}) = [4 - 4\cos(0.2\pi) + 2\cos(0.4\pi)]e^{-j2(0.2\pi)} \qquad M=4$$

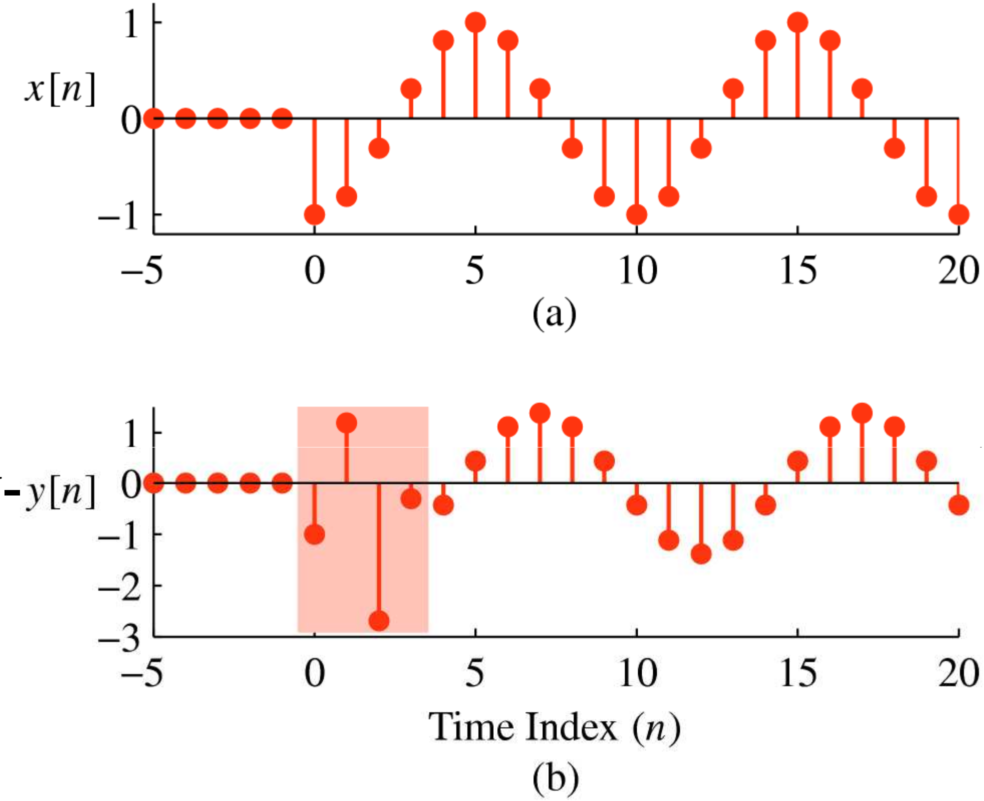
- The steady-state output is

$$y[n] = 1.382\cos(0.2\pi(n-2) - \pi) \qquad 4 \le n$$

- The frequency response has allowed us to find a simple expression for the output everywhere in the steady-state region.

Example

- *M*=4
- The transient region is $0 \le n \le 3$
- The steady-state region is $n \ge 4$
- The signal in the steady-state region is simply a scaled and shifted version of the input



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Impulse Response vs. Frequency Response

- Compare

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\Updownarrow$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k}$$

- Each term $b_k x[n-k]$ corresponds to a term $b_k e^{-j\hat{\omega}k}$ or $h[k]e^{-j\hat{\omega}k}$
- The frequency response $H(e^{j\hat{\omega}})$ can be determined directly from the impulse response since the impulse response of the FIR system consists of the sequence of filter coefficients.

Impulse Response vs. Frequency Response

Time Domain	⇔	*Frequency Domain*
$h[n] = \sum_{k=0}^{M} h[k]\delta[n-k]$	⇔	$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}$

- The process of going from the difference equation or impulse response to the frequency response is straightforward for the FIR filter.

Example

- Consider the FIR filter defined by the impulse response $h[n] = -\delta[n] + 3\delta[n-1] - \delta[n-2]$
- The filter coefficients is $\{b_k\} = \{-1, 3, -1\}$
- The difference equation is
  $y[n] = -x[n] + 3x[n-1] - x[n-2]$
- The frequency response of this system is
  $H(e^{j\hat{\omega}}) = -1 + 3e^{-j\hat{\omega}} - e^{-j2\hat{\omega}}$

Time Domain	⟺	*Frequency Domain*
$h[n] = \sum_{k=0}^{M} h[k]\delta[n-k]$	⟺	$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}$

Example

- The frequency response is given

  $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(3 - 2\cos\hat{\omega})$

- Since $\cos\hat{\omega} = \frac{1}{2}(e^{j\hat{\omega}} + e^{-j\hat{\omega}})$, we can write

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}\left[3 - 2\left(\frac{e^{j\hat{\omega}} + e^{-j\hat{\omega}}}{2}\right)\right]$$
$$= -1 + 3e^{-j\hat{\omega}} - e^{-j\hat{\omega}2}$$

- Which corresponds to the following FIR difference equation

$$y[n] = -x[n] + 3x[n-1] - x[n-2]$$

Periodicity

- $H(e^{j\hat{\omega}})$ is always a periodic function with period 2π

$$H(e^{j(\hat{\omega}+2\pi)}) = \sum_{k=0}^{M} b_k e^{-j(\hat{\omega}+2\pi)k}$$

$$e^{-j2\pi k} = 1$$

$$= \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} e^{-j2\pi k} = H(e^{j\hat{\omega}})$$

- It's not surprising that it would have this property, since, as we have seen in Chapter 4, a change in the input frequency by 2π is not detectable

$$x[n] = Xe^{j(\hat{\omega}+2\pi)n} = Xe^{j\hat{\omega}n}e^{j2\pi n} = Xe^{j\hat{\omega}n}$$

- It's always sufficient to specify the frequency response only over and interval of one period, i.e. $-\pi < \hat{\omega} \leq \pi$

Conjugate Symmetry

- Conjugate symmetry

  $$H(e^{-j\hat{\omega}}) = H^*(e^{j\hat{\omega}})$$

  which is true whenever the filter coefficients are real so that $b_k = b_k^*$ (equivalently $h[k] = h^*[k]$).
- We can prove this property

  $$H^*(e^{j\hat{\omega}}) = \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right)^*$$
  $$= \sum_{k=0}^{M} b_k^* e^{+j\hat{\omega}k}$$
  $$= \sum_{k=0}^{M} b_k e^{-j(-\hat{\omega})k} = H(e^{-j\hat{\omega}})$$

Conjugate Symmetry

- The conjugate-symmetry property implies that the magnitude function is an even function of $\hat{\omega}$ and the phase is an odd function

$$|H(e^{-j\hat{\omega}})| = |H(e^{j\hat{\omega}})|$$

$$\angle H(e^{-j\hat{\omega}}) = -\angle H(e^{j\hat{\omega}})$$

- Similarly, the real part is an even function of $\hat{\omega}$ and the imaginary part is an odd function

$$\mathfrak{Re}\{H(e^{-j\hat{\omega}})\} = \mathfrak{Re}\{H(e^{j\hat{\omega}})\}$$

$$\mathfrak{Im}\{H(e^{-j\hat{\omega}})\} = -\mathfrak{Im}\{H(e^{j\hat{\omega}})\}$$

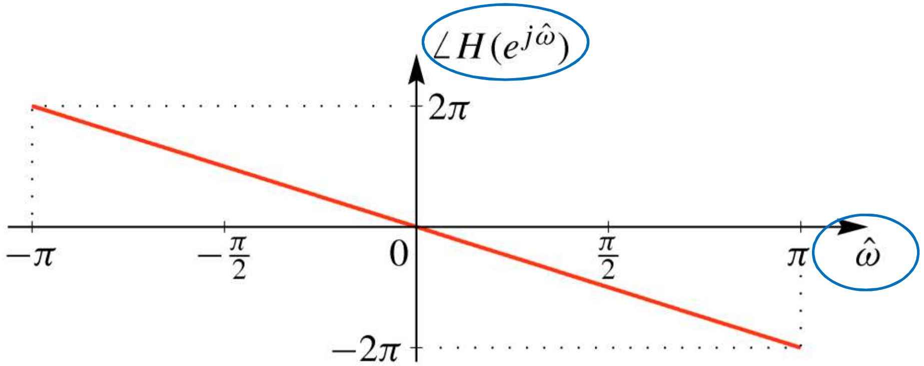
Graphical Representation

- For a given system, the frequency response usually varies with frequency, so that sinusoids of different frequencies are treated differently by the system.
- By appropriate choice of the coefficients, b_k, a wide variety of frequency response shapes can be realized.
- To visualize the variation of the frequency response with frequency, it's useful to plot $H(e^{j\hat{\omega}})$ versus $\hat{\omega}$

Delay System

- The delay system is given by the difference equation
  $y[n] = x[n - n_0]$
- It has only one nonzero filter coefficient, b_{n0}=1, so its frequency response is $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}n_0}$
- The magnitude response is one for all frequencies
- The phase is given by the equation of a straight line with a slope equal to $-n_0$

In the case of $n_0 = 2$



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First-Difference System

- Consider the first-difference system

$$y[n] = x[n] - x[n-1]$$

- The frequency response of this LTI system is

$$H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}} = 1 - \cos\hat{\omega} + j\sin\hat{\omega}$$

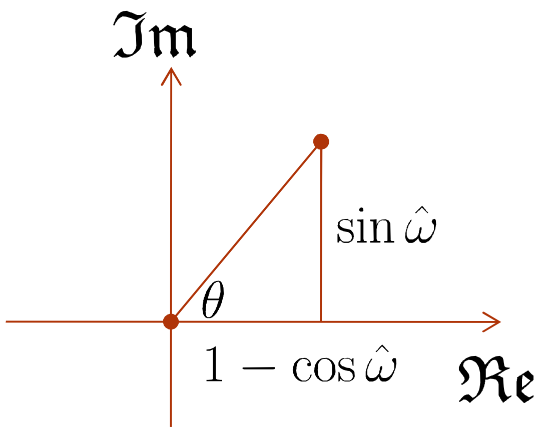
$$\mathfrak{Re}\{H(e^{j\hat{\omega}})\} = 1 - \cos\hat{\omega}$$

$$\mathfrak{Im}\{H(e^{j\hat{\omega}})\} = \sin\hat{\omega}$$

$$|H(e^{j\hat{\omega}})| = [(1-\cos\hat{\omega})^2 + \sin^2\hat{\omega}]^{1/2}$$

$$= [2(1-\cos\hat{\omega})]^{1/2} = 2|\sin(\hat{\omega}/2)|$$

$$\angle H(e^{j\hat{\omega}}) = \arctan\left(\frac{\sin\hat{\omega}}{1-\cos\hat{\omega}}\right)$$

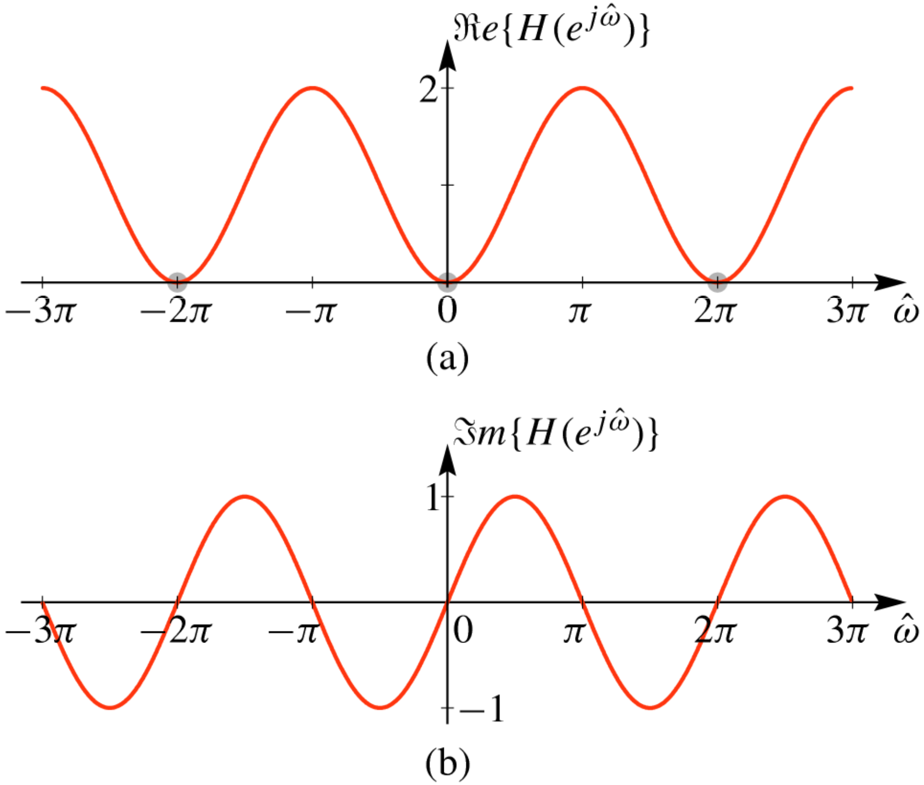


$$\theta = \arctan\left(\frac{\sin\hat{\omega}}{1-\cos\hat{\omega}}\right)$$

First-Difference System

- The plots verify that $H(e^{j\hat{\omega}})$ is periodic with period 2π, and they verify the conjugate symmetry properties

$$\mathfrak{Re}\{H(e^{j\hat{\omega}})\} = 1 - \cos\hat{\omega}$$
$$\mathfrak{Im}\{H(e^{j\hat{\omega}})\} = \sin\hat{\omega}$$



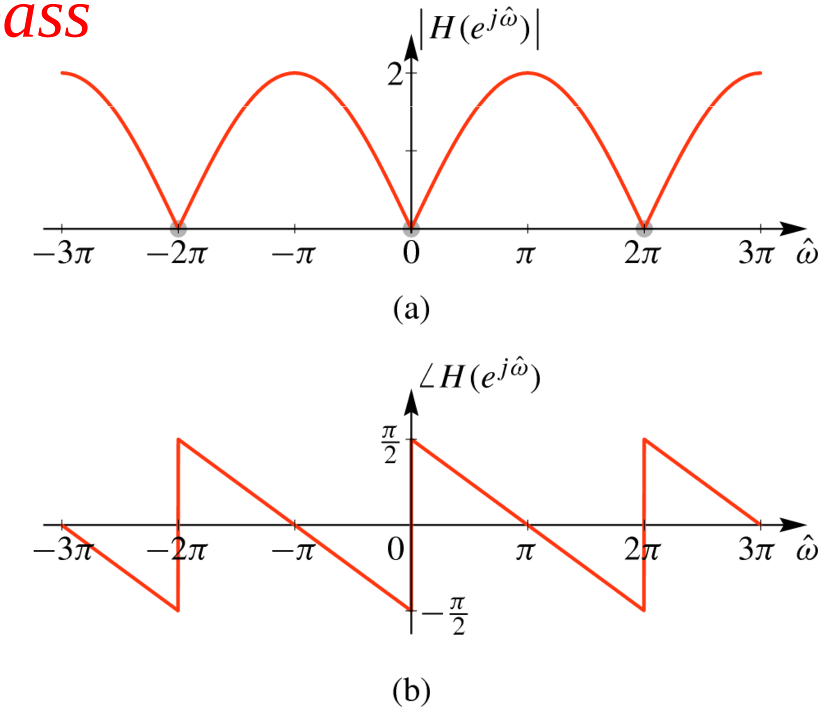
$$\mathfrak{Re}\{H(e^{-j\hat{\omega}})\} = \mathfrak{Re}\{H(e^{j\hat{\omega}})\}$$
$$\mathfrak{Im}\{H(e^{-j\hat{\omega}})\} = -\mathfrak{Im}\{H(e^{j\hat{\omega}})\}$$

First-Difference System

- In this figure, $H(e^{j0}) = 0$, so we can see that the system completely removes components with $\hat{\omega} = 0$
- The system emphasizes the higher frequencies (near $\hat{\omega} = \pi$) relative to the lower frequencies, so it would be called a *highpass filter*.

$$|H(e^{j\hat{\omega}})| = 2|\sin(\hat{\omega}/2)|$$

$$\angle H(e^{j\hat{\omega}}) = \arctan\left(\frac{\sin\hat{\omega}}{1-\cos\hat{\omega}}\right)$$



First-Difference System

- There is a simpler approach for getting the magnitude and phase when the sequence of coefficients is either symmetric or antisymmetric about a central point.

$$b_k = -b_{M-k}$$

- The trick is to factor out an exponential whose phase is half of the filter order (*M*/2) times $\hat{\omega}$, and then use the inverse Euler formula

$$\begin{aligned} H(e^{j\hat{\omega}}) &= 1 - e^{-j\hat{\omega}} \\ &= e^{-j\hat{\omega}/2}(e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}) \\ &= 2je^{-j\hat{\omega}/2}\sin(\hat{\omega}/2) \\ &= 2\sin(\hat{\omega}/2)e^{j(\pi/2-\hat{\omega}/2)} \end{aligned}$$

First-Difference System

- The form derived for $H(e^{j\hat{\omega}})$ is almost a valid polar form, but since $\sin(\hat{\omega}/2)$ is negative for $-\pi < \hat{\omega} < 0$, we must write $|H(e^{j\hat{\omega}})| = 2|\sin(\hat{\omega}/2)|$ and absorb the algebraic sign into the phase response for $-\pi < \hat{\omega} < 0$

$$\angle H(e^{j\hat{\omega}}) = \begin{cases} \pi/2 - \hat{\omega}/2 & 0 < \hat{\omega} < \pi \\ -\pi + \pi/2 - \hat{\omega}/2 & -\pi < \hat{\omega} < 0 \end{cases}$$

$$H(e^{j\hat{\omega}}) = 2\sin(\hat{\omega}/2)e^{j(\pi/2 - \hat{\omega}/2)}$$

- Remember that $-1 = e^{\pm j\pi}$, so we can add either $+\pi$ or $-\pi$ to the phase for $-\pi < \hat{\omega} < 0$. In this case, we add $-\pi$ so that the resulting phase curve remains between $-\pi$ and $+\pi$ radians for all $\hat{\omega}$

Example

- Suppose that the input is $x[n] = 4 + 2\cos(0.3\pi n - \pi/4)$
- The output $y[n] = x[n] - x[n-1]$

$$y[n] = 4 + 2\cos(0.3\pi n - \pi/4) - 4 - 2\cos(0.3\pi(n-1) - \pi/4)$$
$$= 2\cos(0.3\pi n - \pi/4) - 2\cos(0.3\pi n - 0.55\pi)$$

- We see that the first-difference system removes the constant value and leaves two cosine signals of the same frequency.
- Since the first-difference system has frequency response $H(e^{j\hat{\omega}}) = 2\sin(\hat{\omega}/2)e^{j(\pi/2-\hat{\omega}/2)}$, the output is

$$y[n] = 4H(e^{j0}) + 2|H(e^{j0.3\pi})|\cos(0.3\pi n - \pi/4 + \angle H(e^{j0.3\pi}))$$

Example

$$H(e^{j\hat{\omega}}) = 2\sin(\hat{\omega}/2)e^{j(\pi/2-\hat{\omega}/2)}$$

- Since $H(e^{j0}) = 0$ and

  $$H(e^{j0.3\pi}) = 2j\sin(0.3\pi/2)e^{-j0.3\pi/2} = 0.908e^{j(\pi/2-0.15\pi)}$$

- The output will be

$$y[n] = 4H(e^{j0}) + 2|H(e^{j0.3\pi})|\cos(0.3\pi n - \pi/4 + \angle H(e^{j0.3\pi}))$$

⇓

$$y[n] = (0.908)(2)\cos(0.3\pi n - \pi/4 + \pi/2 - 0.3\pi/2)$$
$$= 1.816\cos(0.3\pi n + 0.1\pi)$$

A Simple Lowpass Filter

- In Examples 6-1, 6-3, and 6-4, the system had frequency response

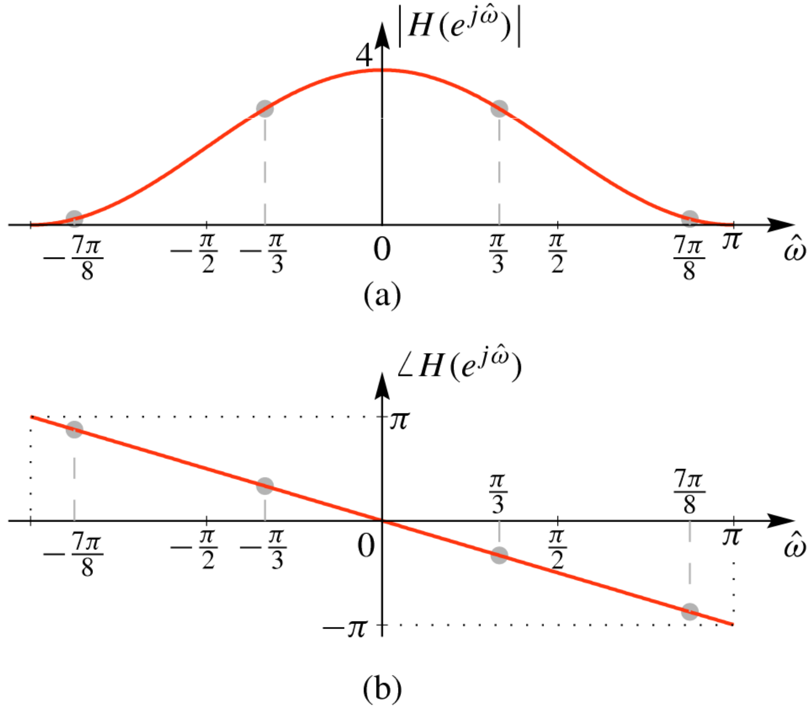
  $$H(e^{j\hat{\omega}}) = 1 + 2e^{-j\hat{\omega}} + e^{-j\hat{\omega}2} = e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega})$$

- Since $(2 + 2\cos\hat{\omega}) \geq 0$ for all $\hat{\omega}$, it follows that

  $$|H(e^{j\hat{\omega}})| = (2 + 2\cos\hat{\omega})$$

  $$\angle H(e^{j\hat{\omega}}) = -\hat{\omega}$$

- It tends to suppress high frequencies (close to $\hat{\omega} = \pi$)
- *Lowpass filter*



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- We evaluate $H(e^{j\hat{\omega}})$ at frequencies 0, $\pi/3$, and $7\pi/8$

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$$H(e^{j7\pi/8}) = 0.1522e^{-j7\pi/8}$$

$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

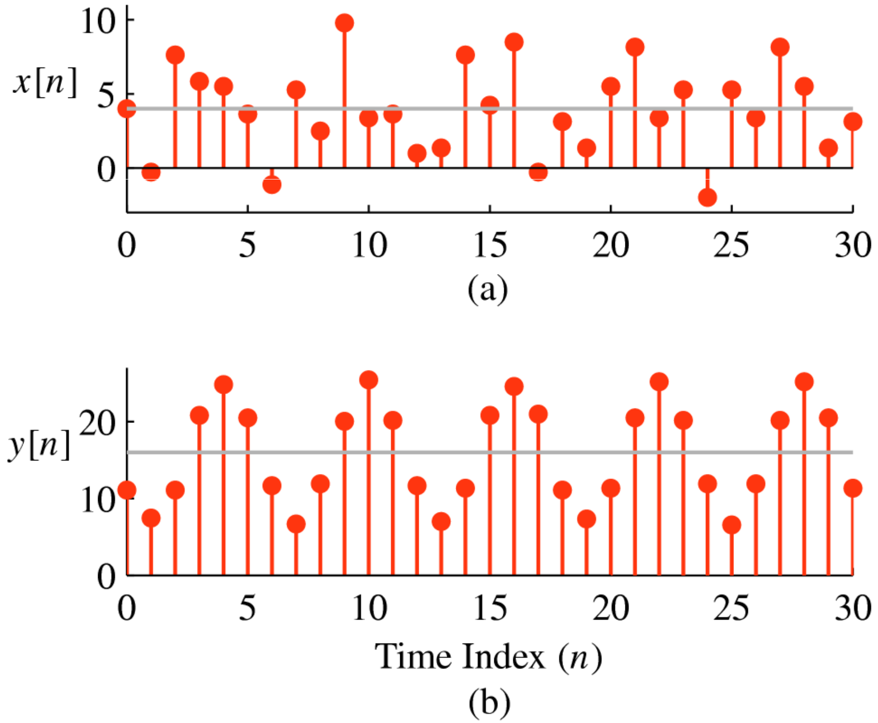
- Output

$$y[n] = 4\cdot 4 + 3\cdot 3\cos\left(\frac{\pi}{3}n - \frac{\pi}{3} - \frac{\pi}{2}\right) + 0.1522\cdot 3\cos\left(\frac{7\pi}{8}n - \frac{7\pi}{8}\right)$$
$$= 16 + 9\cos\left(\frac{\pi}{3}(n-1) - \frac{\pi}{2}\right) + 0.4567\cos\left(\frac{7\pi}{8}(n-1)\right)$$

Example

$$x[n] = 4 + 3\cos\left(\frac{\pi}{3}n - \frac{\pi}{2}\right) + 3\cos\left(\frac{7\pi}{8}n\right)$$
$$y[n] = 16 + 9\cos\left(\frac{\pi}{3}(n-1) - \frac{\pi}{2}\right) + 0.4567\cos\left(\frac{7\pi}{8}(n-1)\right)$$

The DC component is indicated as a gray horizontal line.



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