Lecture 17: 4.9
Matrix Transformations

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Zero Transformation

- Zero Transformation from R^n to R^m
  - If 0 is the $m \times n$ zero matrix and **0** is the zero vector in R^n, then for every vector **x** in R^n

$$T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$$

  - So multiplication by zero maps every vector in R^n into the zero vector in R^m. We call T_0 the zero transformation from R^n to R^m.

Identity Operator

- Identity Operator on R^n
  - If I is the $n \times n$ identity, then for every vector $\mathbf{x}$ in R^n

$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

  - So multiplication by I maps every vector in R^n into itself.
  - We call T_I the identity operator on R^n.

A Procedure for Finding Standard Matrices

- To find the standard matrix A for a matrix transformations from R^n to R^m:
- $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for R^n.
- Suppose that the images of these vectors under the transformation T_A are

$$T_A(\mathbf{e}_1) = A\mathbf{e}_1,\ T_A(\mathbf{e}_2) = A\mathbf{e}_2,\ \ldots,\ T_A(\mathbf{e}_n) = A\mathbf{e}_n$$

- $A\mathbf{e}_j$ is just the jth column of the matrix A, Thus,

$$A = [T] = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \ldots \mid T(\mathbf{e}_n)]$$

Reflection Operators

- In general, operators on R^2 and R^3 that map each vector into its symmetric image about some line or plane are called reflection operators.
- Such operators are linear.

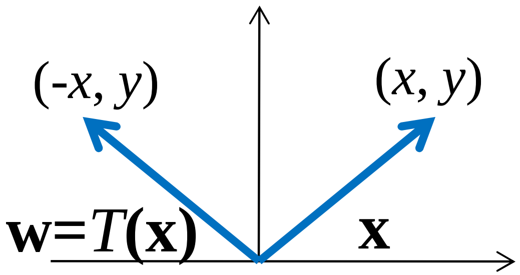
Example

- If we let $\mathbf{w}=T(\mathbf{x})$, then the equations relating the components of $\mathbf{x}$ and $\mathbf{w}$ are

$$w_1 = -x = -x + 0y$$
$$w_2 = y = 0x + y$$

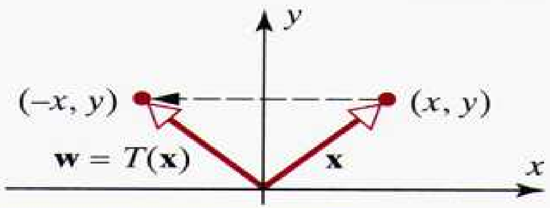
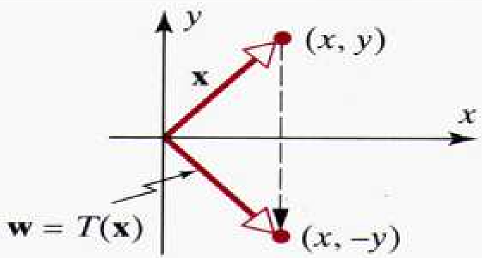
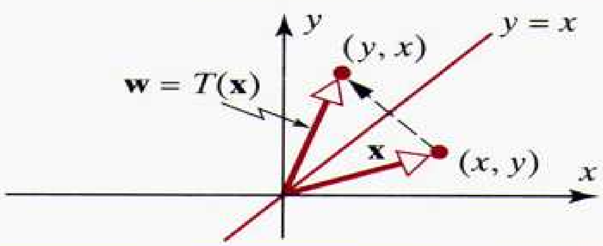
or, in matrix form

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

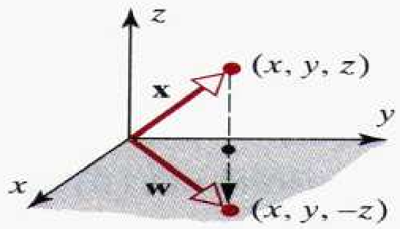
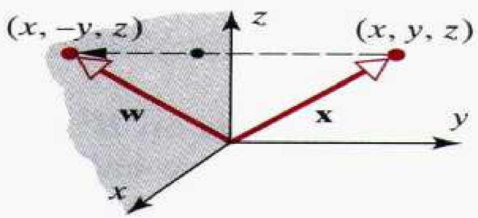
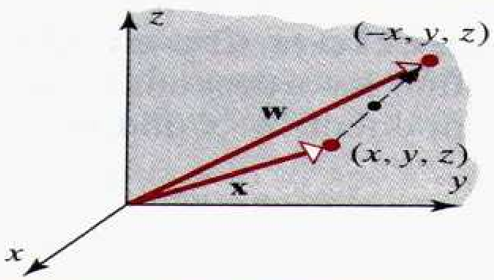


- The standard matrix for T is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Reflection Operators (2-Space)

Operator	Illustration	Equations	Standard Matrix
Reflection about the y-axis		$w_1 = -x$ $w_2 = y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the x-axis		$w_1 = x$ $w_2 = -y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the line $y = x$		$w_1 = y$ $w_2 = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Reflection Operators (3-Space)

Operator	Illustration	Equations	Standard Matrix
Reflection about the xy-plane		$w_1 = x$ $w_2 = y$ $w_3 = -z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane		$w_1 = x$ $w_2 = -y$ $w_3 = z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane		$w_1 = -x$ $w_2 = y$ $w_3 = z$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Projection Operators

- In general, a projection operator (or more precisely an orthogonal projection operator) on R^2 or R^3 is any operator that maps each vector into its orthogonal projection on a line or plane through the origin.
- The projection operators are linear.

Example

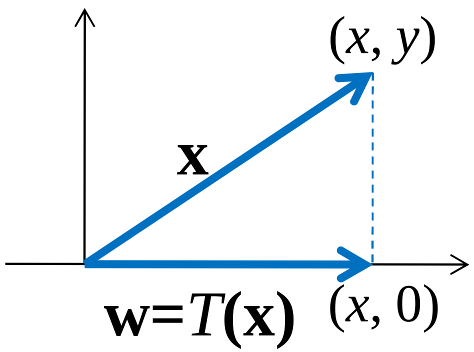
- Consider the operator $T: R^2 \longrightarrow R^2$ that maps each vector into its orthogonal projection on the x-axis. The equations relating the components of $\mathbf{x}$ and $\mathbf{w}=T(\mathbf{x})$ are

$$w_1 = x = x + 0y$$
$$w_2 = 0 = 0x + 0y$$

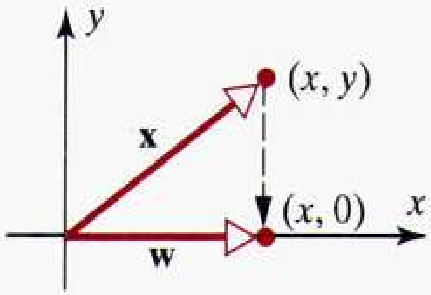
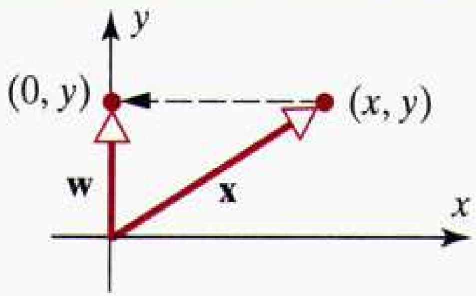
or, in matrix form

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

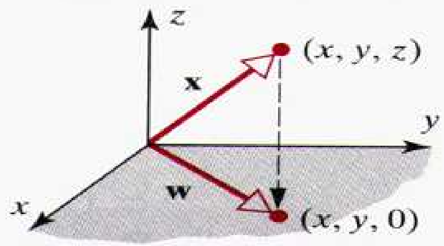
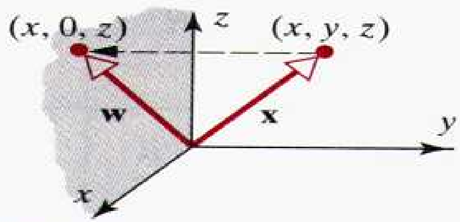
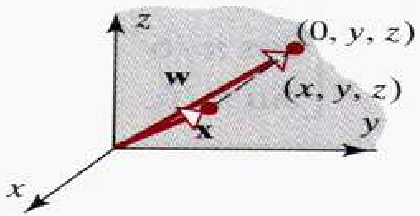
- The standard matrix for T is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$



Projection Operators

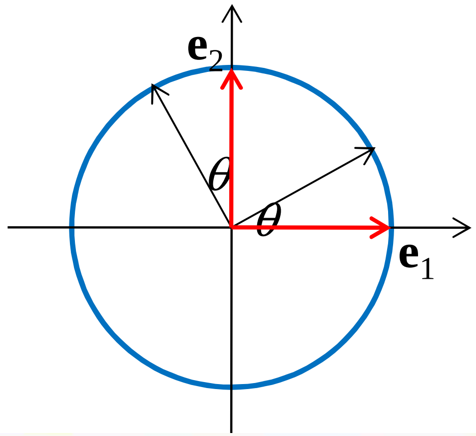
Operator	Illustration	Equations	Standard Matrix
Orthogonal projection on the x-axis		$w_1 = x$ $w_2 = 0$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection on the y-axis		$w_1 = 0$ $w_2 = y$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

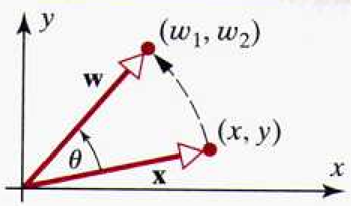
Projection Operators

Operator	Illustration	Equations	Standard Matrix
Orthogonal projection on the xy-plane	 	$w_1 = x$ $w_2 = y$ $w_3 = 0$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection on the xz-plane	 	$w_1 = x$ $w_2 = 0$ $w_3 = z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection on the yz-plane	 	$w_1 = 0$ $w_2 = y$ $w_3 = z$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotation Operators

- The rotation operator $T:R^2 \rightarrow R^2$ moves points counterclockwise about the origin through an angle θ
- Find the standard matrix
- $T(\mathbf{e}_1) = T(1,0) = (\cos\theta, \sin\theta)$
- $T(\mathbf{e}_2) = T(0,1) = (-\sin\theta, \cos\theta)$



Operator	Illustration	Equations	Standard Matrix
Rotation through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Example

- If each vector in R^2 is rotated through an angle of π/6 (30°) ,then the image **w** of a vector

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

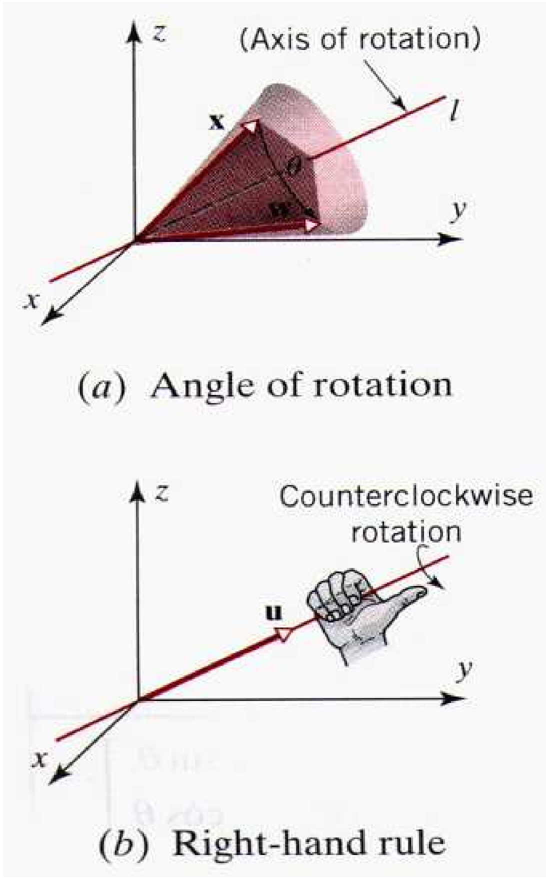
is $$\mathbf{w} = \begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2\, x - 1/2\, y \\ 1/2\, x + \sqrt{3}/2\, y \end{bmatrix}$$

- For example, the image of the vector

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{is} \quad \mathbf{w} = \begin{bmatrix} \dfrac{\sqrt{3} - 1}{2} \\ \dfrac{1 + \sqrt{3}}{2} \end{bmatrix}$$

A Rotation of Vectors in R^3

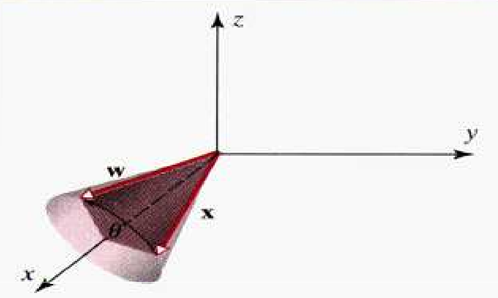
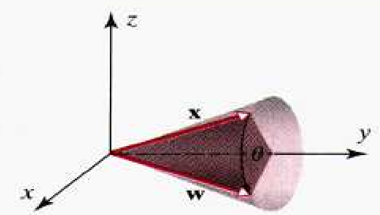
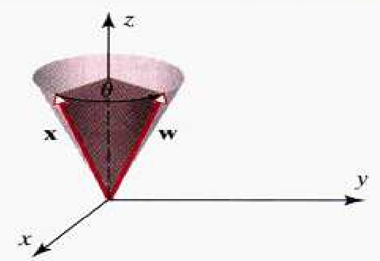
- A rotation of vectors in R^3 is usually described in relation to a ray emanating from the origin, called the axis of rotation.
- As a vector revolves around the axis of rotation it sweeps out some portion of a cone.
- The angle of rotation is described as "clockwise" or "counterclockwise" in relation to a viewpoint that is along the axis of rotation *looking toward the origin.*
- The axis of rotation can be specified by a nonzero vector **u** that runs along the axis of rotation and has its initial point at the origin.
- The counterclockwise direction for a rotation about its axis can be determined by a "right-hand rule".



(*a*) Angle of rotation

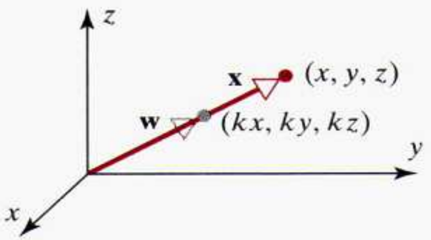
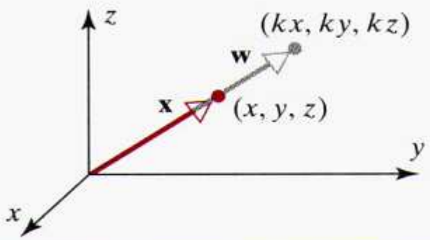
(*b*) Right-hand rule

A Rotation of Vectors in R^3

Operator	Illustration	Equations	Standard Matrix
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Dilation and Contraction Operators

- If k is a nonnegative scalar, the operator on R^2 or R^3 is called a contraction with factor k if $0 \le k \le 1$ (以因素k收縮) and a dilation with factor k if $k \ge 1$ (以因素k膨脹).

Operator	Illustration	Equations	Standard Matrix
Contraction with factor k on R^3 $(0 \le k \le 1)$		$w_1 = kx$ $w_2 = ky$ $w_3 = kz$	$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k on R^3 $(k \ge 1)$		$w_1 = kx$ $w_2 = ky$ $w_3 = kz$	

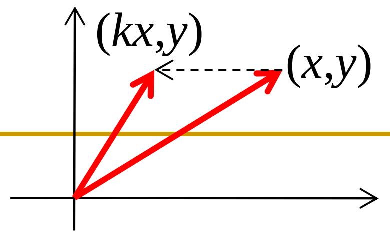
Compression or Expansion

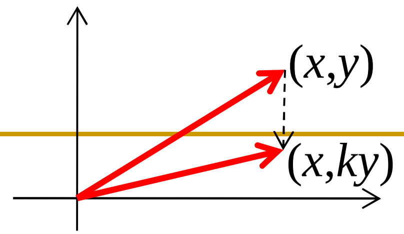
- If $T: R^2 \to R^2$ is a compression ($0<k<1$) or expansion ($k>1$) in the x-direction with factor k, then

$$T(\boldsymbol{e}_1) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}k\\0\end{bmatrix} \qquad T(\boldsymbol{e}_2) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

so the standard matrix for T is $\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$.

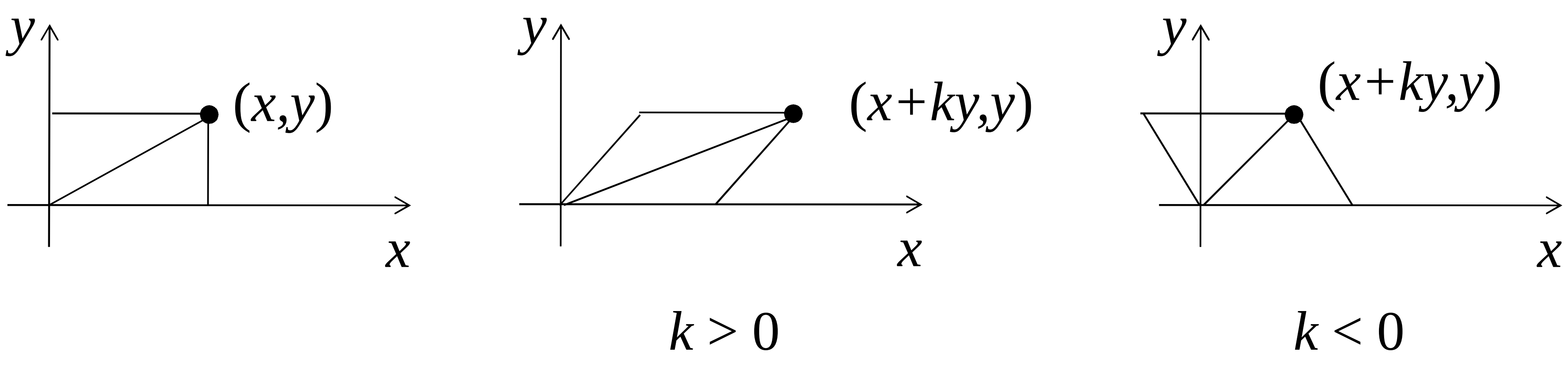
- Similarly, the standard matrix for a compression or expansion in the y-direction is $\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$





Shears

- **A shear (剪) in the *x*-direction with factor *k*** is a transformation that moves each point (x,y) parallel to the x-axis by an amount ky to the new position $(x+ky,y)$.
- Points farther from the x-axis move a greater distance than those closer.



Shears

- If $T: R^2 \rightarrow R^2$ is a shear with factor k in the x-direction, then

$$T(\boldsymbol{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x + ky \\ y \end{bmatrix} = \begin{bmatrix} 1 + k0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

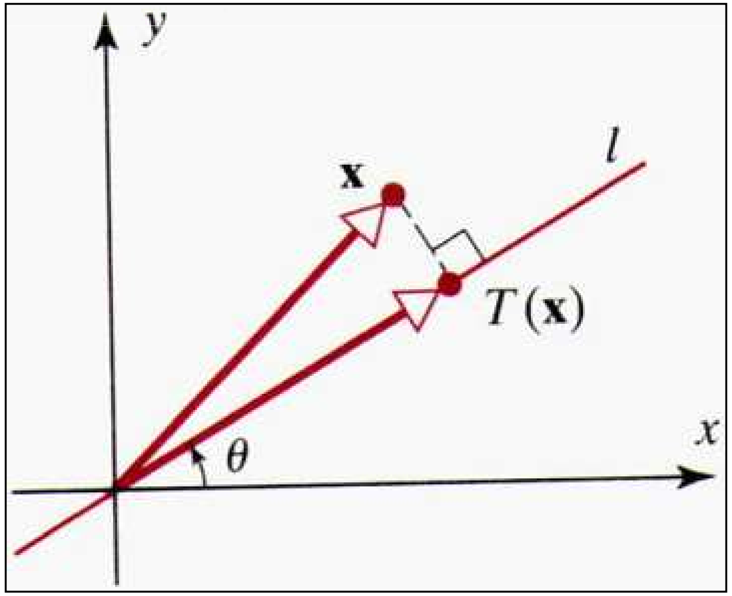
$$T(\boldsymbol{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} x + ky \\ y \end{bmatrix} = \begin{bmatrix} 0 + k1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

- The standard matrix for T is $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

- Similarly, the standard matrix for a shear in the y-direction with factor k is $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Example (Standard Matrix for a Projection Operator)

- Let l be the line in the xy-plane that passes through the origin and makes an angle θ with the positive x-axis, where $0 \leq \theta \leq \pi$. Let $T: R^2 \rightarrow R^2$ be a linear operator that maps each vector into orthogonal projection on l.
  - Find the standard matrix for T.
  - Find the orthogonal projection of the vector $\mathbf{x} = (1,5)$ onto the line through the origin that makes an angle of $\theta = \pi/6$ with the positive x-axis.



Example

- The standard matrix for T can be written as

$$[T] = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)]$$

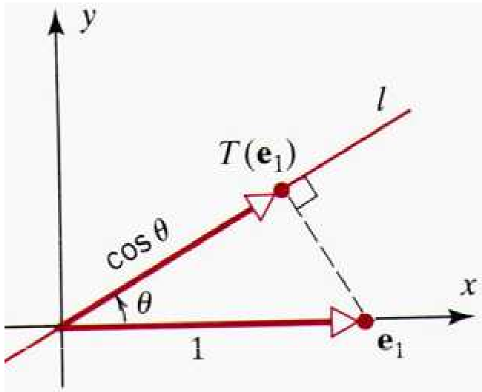
- Consider the case $0 \le \theta \le \pi/2$.
  - $\|T(\mathbf{e}_1)\| = \cos\theta$

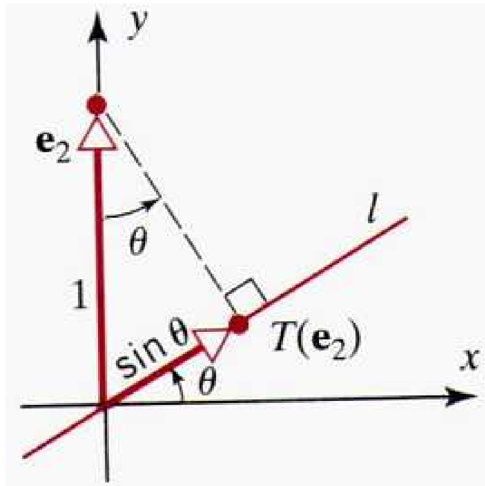
$$\Rightarrow \quad T(\mathbf{e}_1) = \begin{bmatrix} \|T(\mathbf{e}_1)\|\cos\theta \\ \|T(\mathbf{e}_1)\|\sin\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix}$$

  - $\|T(\mathbf{e}_2)\| = \sin\theta$

$$\Rightarrow \quad T(\mathbf{e}_2) = \begin{bmatrix} \|T(\mathbf{e}_2)\| = \cos\theta \\ \|T(\mathbf{e}_2)\| = \sin\theta \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$





Example

$$[T] = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

- Since sin (π/6) = 1/2 and cos (π/6) = $\sqrt{3}$/2, it follows from part (a) that the standard matrix for this projection operator is

$$[T] = \begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix}$$

Thus,

$$T\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \dfrac{3+5\sqrt{3}}{4} \\ \dfrac{\sqrt{3}+5}{4} \end{bmatrix}$$

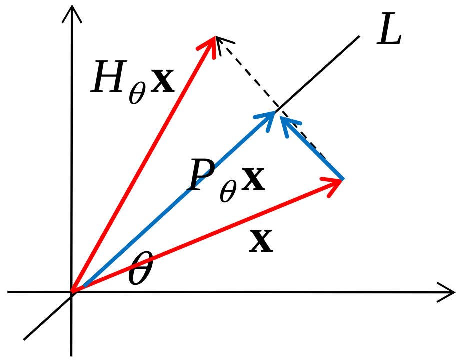
Reflections About Lines Through the Origin

- Let P_θ denote the standard matrix of orthogonal projections on lines through the origin

$P_\theta \mathbf{x} - \mathbf{x} = (1/2)(H_\theta \mathbf{x} - \mathbf{x})$, or equivalently $H_\theta \mathbf{x} = (2 P_\theta - I)\mathbf{x}$

- $H_\theta = (2 P_\theta - I)$

$$H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$



Lecture 17: 4.10 Properties of Matrix Transformations

Wei-Ta Chu

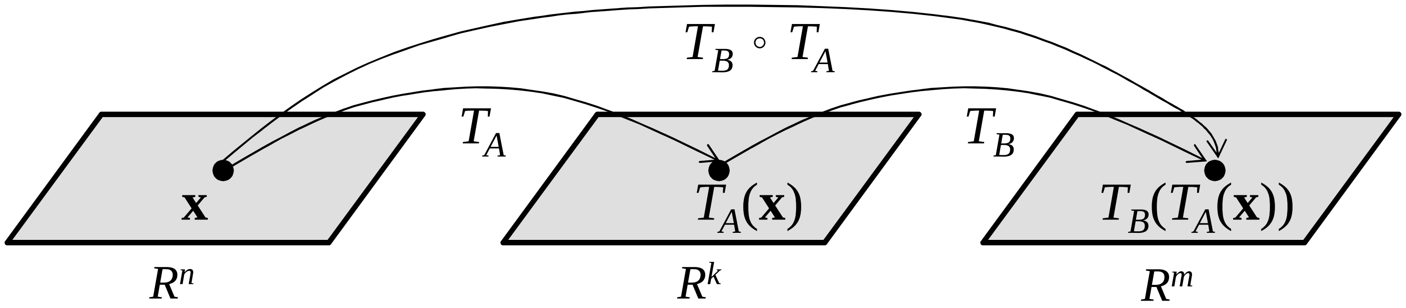
2011/11/16

Composition of T_B with T_A

- Definition
  - If $T_A : R^n \to R^k$ and $T_B : R^k \to R^m$ are linear transformations, *the composition of T_B with T_A*, denoted by $T_B \circ T_A$ (read "T_B circle T_A"), is the function defined by the formula

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))$$

where $\mathbf{x}$ is a vector in R^n.



Composition of T_B with T_A

- This composition is itself a matrix transformation since

  $$(T_B \circ T_A)(\mathbf{x})=(T_B(T_A(\mathbf{x}))=B(T_A(\mathbf{x}))=B(A\mathbf{x})=(BA)\mathbf{x}$$

- It is multiplication by BA, i.e. $T_B \circ T_A = T_{BA}$
- The compositions can be defined for more than two linear transformations.
- For example, if $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$,and $T_3 : W \rightarrow Y$ are linear transformations, then the composition $T_3 \circ T_2 \circ T_1$ is defined by $(T_3 \circ T_2 \circ T_1)(\mathbf{u}) = T_3(T_2(T_1(\mathbf{u})))$

Remark

- It is not true, in general, that $AB = BA$
- So it is not true, in general, that $T_B \circ T_A = T_A \circ T_B$

Example

- Let $T_1 : R^2 \to R^2$ and $T_2 : R^2 \to R^2$ be the matrix operators that rotate vectors through the angles θ_1 and θ_2, respectively.
- The operation $(T_2 \circ T_1)(\mathbf{x}) = T_2(T_1(\mathbf{x}))$ first rotates $\mathbf{x}$ through the angle θ_1, then rotates $T_1(\mathbf{x})$ through the angle θ_2.

$$[T_1] = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \qquad [T_2] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$[T_2 \circ T_1] = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$

$$\begin{aligned} [T_2][T_1] &= \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -(\cos\theta_2\sin\theta_1 + \sin\theta_2\cos\theta_1) \\ \sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1 & -\sin\theta_2\sin\theta_1 + \cos\theta_2\cos\theta_1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix} = [T_2] \circ [T_1] \end{aligned}$$

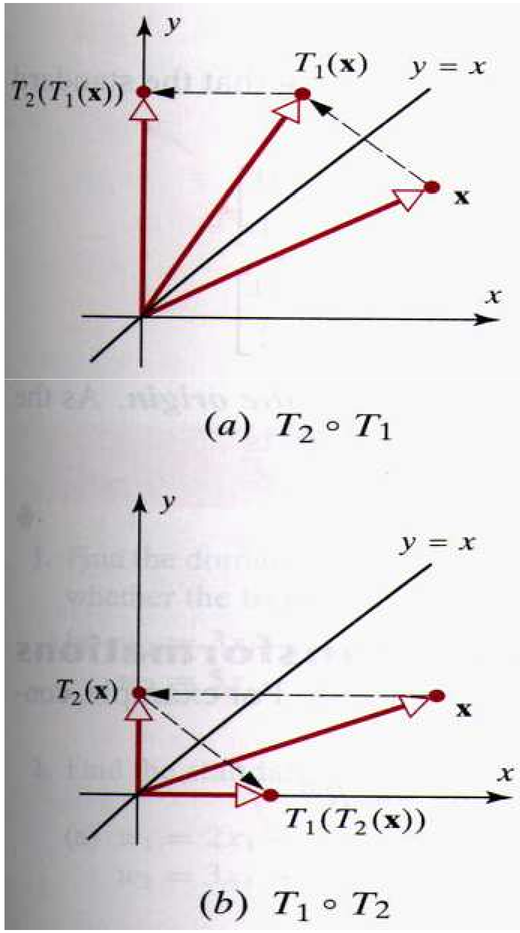
Composition is Not Commutative

- Let T_1 be the reflection operator
- Let T_2 be the orthogonal projection on the y-axis

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

so $[T_1 \circ T_2] \neq [T_2 \circ T_1]$



Composition of Two Reflections

- Let T_1 be the reflection about the y-axis, and let T_2 be the reflection about the x-axis. In this case, $T_1 \circ T_2$ and $T_2 \circ T_1$ are the same.

$$(T_1 \circ T_2)(x, y) = T_1(x, -y) = (-x, -y)$$

$$(T_2 \circ T_1)(x, y) = T_2(-x, y) = (-x, -y)$$

$$[T_1 \circ T_2] = [T_1]\,[T_2] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[T_2 \circ T_1] = [T_2]\,[T_1] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

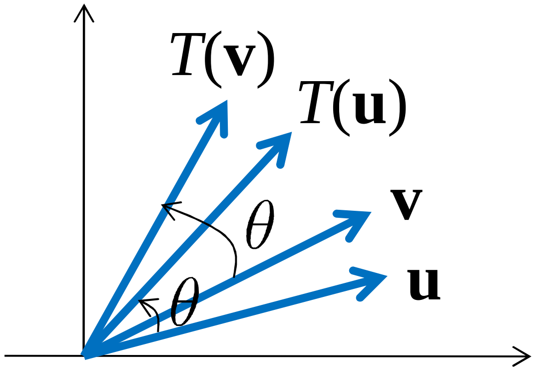
One-to-One Linear transformations

- Definition
  - A linear transformation $T : R^n \to R^m$ is said to be one-to-one if T maps distinct vectors (points) in R^n into distinct vectors (points) in R^m

- Remark:
  - That is, for each vector **w** in the range of a one-to-one linear transformation T, there is exactly one vector **x** such that $T(\mathbf{x}) = \mathbf{w}$.

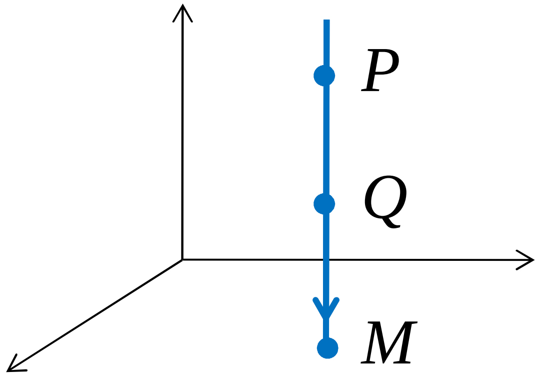
Example

One-to-one linear transformation



Distinct vectors **u** and **v** are rotated into distinct vectors $T(\mathbf{u})$ and $T(\mathbf{v})$.

Not one-to-one linear transformation



The distinct points P and Q are mapped into the same point M.