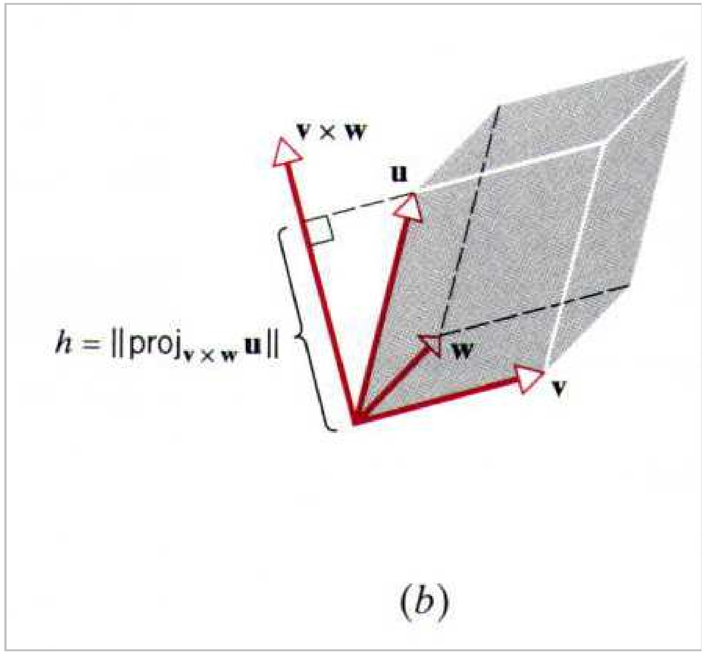
Proof of Theorem 3.5.4(b)



- The area of the base is $\|\boldsymbol{v} \times \boldsymbol{w}\|$
- The height *h* of the parallelepiped is the length of the orthogonal projection of **u** on $\boldsymbol{v} \times \boldsymbol{w}$

$$h = \|proj_{\boldsymbol{v}\times\boldsymbol{w}}\boldsymbol{u}\| = \frac{|\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})|}{\|\boldsymbol{v}\times\boldsymbol{w}\|}$$

- The volume *V* of the parallelepiped is

$$V = \|\boldsymbol{v}\times\boldsymbol{w}\|\frac{|\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})|}{\|\boldsymbol{v}\times\boldsymbol{w}\|} = |\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})|$$

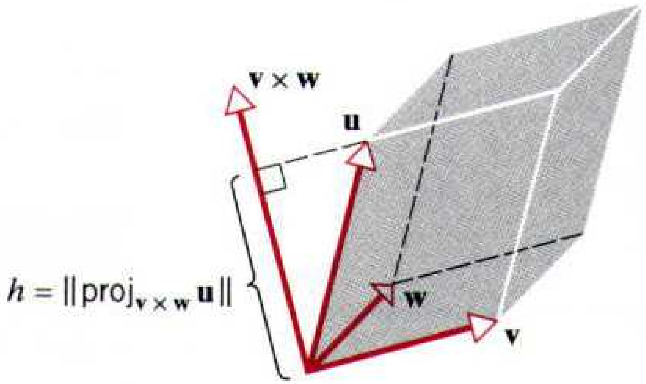
$$\boxed{\|proj_{\boldsymbol{a}}\boldsymbol{u}\| = \frac{|\boldsymbol{u}\cdot\boldsymbol{a}|}{\|\boldsymbol{a}\|}}$$

Remark

$$V = \left| \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right|$$

$$V = \begin{bmatrix} \text{volume of parallelepiped} \\ \text{determined by } \mathbf{u}, \mathbf{v}, \text{ and } \mathbf{w} \end{bmatrix} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



(*b*)

Remark

$V = |\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|$

- We can conclude that

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \pm V$$

where + or – results depending on whether **u** makes an acute or an obtuse angle with $\boldsymbol{v} \times \boldsymbol{w}$

Theorem 3.5.5

- If the vectors **u** = (u_1, u_2, u_3), **v** = (v_1, v_2, v_3), and **w** = (w_1, w_2, w_3) have the same initial point, then they lie in the same plane if and only if

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}=0$$

Lecture 10: 4.1
General Vector Spaces

Wei-Ta Chu

2011/10/19

Definition (Vector Space)

- Let *V* be an arbitrary nonempty set of objects on which two operations are defined:
    - Addition
    - Multiplication by scalars
- If the following *axioms* (公理) are satisfied by all objects **u**, **v**, **w** in *V* and all scalars *k* and *l*, then we call *V* a vector space (向量空間) and we call the objects in *V* vectors.

Definition (Vector Space)

1. If **u** and **v** are objects in *V*, then **u** + **v** is in *V*.
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. There is an object **0** in *V*, called a zero vector for *V*, such that **0** + **u** = **u** + **0** = **u** for all **u** in *V*.
5. For each **u** in *V*, there is an object **-u** in *V*, called a negative of **u**, such that **u** + (**-u**) = (**-u**) + **u** = **0**.
6. If *k* is any scalar and **u** is any object in *V*, then *k***u** is in *V*.
7. *k* (**u** + **v**) = *k***u** + *k***v**
8. (*k* + *l*) **u** = *k***u** + *l***u**
9. *k* (*l***u**) = (*kl*) (**u**)
10. 1**u** = **u**

Remarks

- Depending on the application, *scalars* may be real numbers or complex numbers.
  - Vector spaces in which the scalars are complex numbers are called complex vector spaces (複數向量空間), and those in which the scalars must be real are called real vector spaces (實數向量空間).
- The definition of a vector space specifies neither the nature of the vectors nor the operations.
  - *Any kind of object can be a vector*, and the operations of addition and scalar multiplication may not have any relationship or similarity to the standard vector operations on R^n.
  - *The only requirement is that the ten vector space axioms be satisfied.*

Show a Set as a Vector Space

- Step 1: Identify the set *V* of objects that will become vectors.
- Step 2: Identify the addition and scalar multiplication operations on *V*.
- Step 3. Verify Axioms 1 and 6. Axiom 1 is called ***closure under addition*** (加法封閉性), and Axiom 6 is call ***closure under scalar multiplication*** (純量乘法封閉性).
- Step 4: Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

Example: The Zero Vector Space

- Let V consist of a single object, which we denote by $\mathbf{0}$, and define $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $k\mathbf{0} = \mathbf{0}$ for all scalars k.
- It's easy to check that all the vector space axioms are satisfied.
- We call this the ***zero vector space***.

Example (R^n Is a Vector Space)

- The set $V = R^n$ with the standard operations of addition and scalar multiplication is a vector space.
- Axioms 1 and 6 follow from the definitions of the standard operations on R^n; the remaining axioms follow from Theorem 3.1.1.
- The three most important special cases of R^n are R (the real numbers), R^2 (the vectors in the plane), and R^3 (the vectors in 3-space).

Example (2×2 Matrices)

- Show that the set V of all 2×2 matrices with real entries is a vector space <u>if vector addition is defined to be matrix addition and vector scalar multiplication is defined to be matrix scalar multiplication</u>.
- Let $\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$
- To prove Axiom 1, we must show that **u** + **v** is an object in V; that is, we must show that **u** + **v** is a 2×2 matrix.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

Example

- Similarly, Axiom 6 hold because for any real number k we have

$$k\mathbf{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

  so that $k\mathbf{u}$ is a 2×2 matrix and consequently is an object in V.
- Axioms 2 follows from Theorem 1.4.1a since

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{v}+\mathbf{u}$$

- Similarly, Axiom 3 follows from part (b) of that theorem; and Axioms 7, 8, and 9 follow from part (h), (j), and (e), respectively.

Example

- To prove Axiom 4, let $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

  Then
  $$\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$
  Similarly, **u** + **0** = **u**.
- To prove Axiom 5, let $-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$

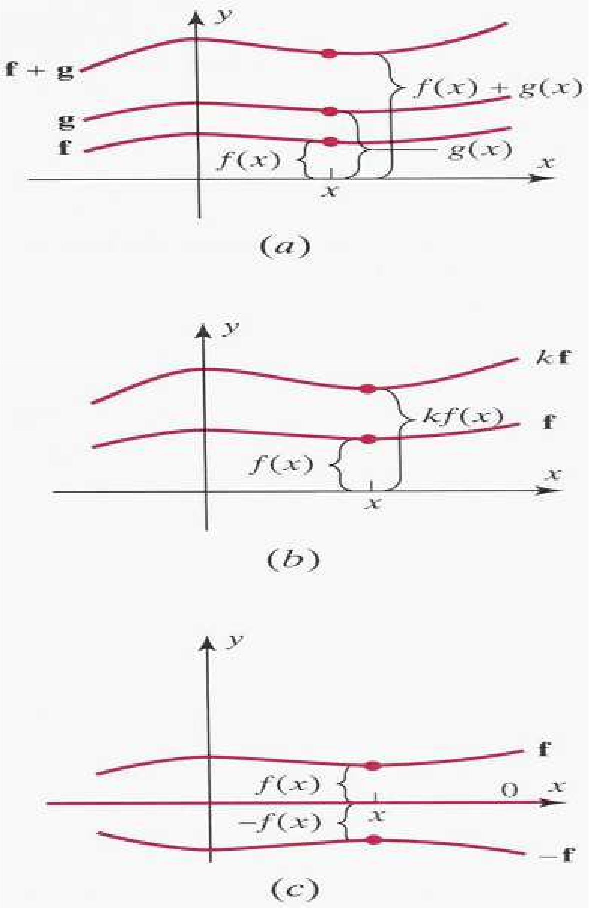
  Then
  $$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$
  Similarly, (-**u**) + **u** = **0**.
- For Axiom 10, 1**u** = **u**.

Example (Vector Space of $m \times n$ Matrices)

- The previous example is a special case of a more general class of vector spaces.
- The arguments in that example can be adapted to show that the set V of all $m \times n$ matrices with real entries, together with the operations <u>matrix addition and scalar multiplication</u>, is a vector space.
- The $m \times n$ zero matrix is the zero vector $\mathbf{0}$, and if $\mathbf{u}$ is the $m \times n$ matrix U, then matrix $-U$ is the negative $-\mathbf{u}$ of the vector $\mathbf{u}$.
- We shall denote this vector space by the symbol M_{mn}

Example (Vector Space of Real-Valued Functions)

- Let V be the set of real-valued functions defined on the entire real line $(-\infty,\infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two such functions and k is any real number, defined the sum function $\mathbf{f} + \mathbf{g}$ and the scalar multiple $k\,\mathbf{f}$, respectively, by $(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$ and $(k\,\mathbf{f})(x)=kf(x)$.
- In other words, the value of the function $\mathbf{f} + \mathbf{g}$ at x is obtained by adding together the values of $\mathbf{f}$ and $\mathbf{g}$ at x (Figure 4.1.1 a). Similarly, the value of $k\,\mathbf{f}$ at x is k times the value of $\mathbf{f}$ at x (Figure 4.1.1 b). This vector space is denoted by $F(-\infty,\infty)$. If $\mathbf{f}$ and $\mathbf{g}$ are vectors in this space, then to say that $\mathbf{f} = \mathbf{g}$ is equivalent to saying that $f(x) = g(x)$ for all x in the interval $(-\infty,\infty)$.
- The vector $\mathbf{0}$ in $F(-\infty,\infty)$ is the constant function that identically zero for all value of x. The negative of a vector $\mathbf{f}$ is the function $-\mathbf{f} = -f(x)$. Geometrically, the graph of $-\mathbf{f}$ is the reflection of the graph of $\mathbf{f}$ across the x-axis (Figure 4.1.1.c).



Example (Not a Vector Space)

- Let $V = R^2$ and define addition and scalar multiplication operations as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

  $$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

  and if k is any real number, then define

  $$k\,\mathbf{u} = (k\,u_1, 0)$$

- There are values of $\mathbf{u}$ for which Axiom 10 fails to hold. For example, if $\mathbf{u} = (u_1, u_2)$ is such that $u_2 \neq 0$,then

  $$1\mathbf{u} = 1\,(u_1, u_2) = (1\,u_1, 0) = (u_1, 0) \neq \mathbf{u}$$

- Thus, V is *not* a vector space with the stated operations.

An Unusual Vector Space

- Let V be the set of positive real numbers, and define the operation on V to be

$$u+v = uv, \; ku = u^k$$

- For example: $1+1 = 1$ and $2(1) = 1^2 = 1$
- The set V with these operations satisfies the 10 vector space axioms and hence is a vector space!
  - Axiom 4: the zero vector in this space is the number 1 since $u+1=u$
  - Axiom 5: the negative of a vector u is its reciprocal ($-u = 1/u$) since $u+(1/u)=u(1/u) = 1 = \mathbf{0}$
  - Axiom 7: $k(u+v) = (uv)^k = u^k v^k = (ku) + (kv)$

Every Plane Through the Origin Is a Vector Space

- *Check all the axioms!*
  - Let V be any plane through the origin in R^3. Since R^3 itself is a vector space, Axioms 2, 3, 7, 8, 9, and 10 hold for all points in R^3 and consequently for all points in the plane V.
  - We need only show that Axioms 1, 4, 5, and 6 are satisfied.

Every Plane Through the Origin Is a Vector Space

- Since the plane V passes through the origin, it has an equation of the form $ax + by + cz = 0$. If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are points in V, then $au_1 + bu_2 + cu_3 = 0$ and $av_1 + bv_2 + cv_3 = 0$. Adding these equations gives $a(u_1 + v_1) + b(u_2 + v_2) + c(u_3 + v_3) = 0$.
- Axiom 1: $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$; thus $\mathbf{u} + \mathbf{v}$ lies in the plane V.
- Axioms 5: Multiplying $au_1 + bu_2 + cu_3 = 0$ through by -1 gives $a(-u_1) + b(-u_2) + c(-u_3) = 0$; thus, $-\mathbf{u} = (-u_1, -u_2, -u_3)$ lies in V.

Theorem 4.1.1

- Let V be a vector space, $\mathbf{u}$ be a vector in V, and k a scalar; then:
    - $0\,\mathbf{u} = \mathbf{0}$
    - $k\,\mathbf{0} = \mathbf{0}$
    - $(-1)\,\mathbf{u} = -\mathbf{u}$
    - If $k\,\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Proof of Theorem 4.1.1(a)

- We can write

$$0\mathbf{u} + 0\mathbf{u} = (0+0)\mathbf{u} \text{ [Axiom 8]}$$

$$= 0\mathbf{u} \text{ [Property of the number 0]}$$

- By Axiom 5 the vector $0\mathbf{u}$ has a negative, $-0\mathbf{u}$. Adding this negative to both sides above yields

$$[0\mathbf{u} + 0\mathbf{u}] + (-0\mathbf{u}) = (0+0)\mathbf{u} + (-0\mathbf{u}) \text{ [Axiom 3]}$$

$$0\mathbf{u} + \mathbf{0} = \mathbf{0} \text{ [Axiom 5]}$$

$$0\mathbf{u} = \mathbf{0} \text{ [Axiom 4]}$$

Proof of Theorem 4.1.1(c)

- To show that (-1)**u** = -**u**, we must demonstrate that **u** + (-1)**u** = **0**.
- To see this, we observe that

$$\begin{aligned} \mathbf{u} + (-1)\mathbf{u} &= 1\mathbf{u} + (-1)\mathbf{u} \quad \text{[Axiom 10]} \\ &= (1 + (-1))\mathbf{u} \quad \text{[Axiom 8]} \\ &= 0\mathbf{u} \quad \text{[Property of numbers]} \\ &= \mathbf{0} \quad \text{[Property (a) above]} \end{aligned}$$

Lecture 10: 4.2
Subspaces

Wei-Ta Chu

2011/10/19

Subspaces (子空間)

- Definition
  - A subset *W* of a vector space *V* is called a subspace of *V* if *W* is itself a vector space under the addition and scalar multiplication defined on *V*.
- In general, one must verify the ten vector space axioms to show that a set *W* with addition and scalar multiplication forms a vector space.
- However, some axioms are inherited from *V*. For example, there is no need to check Axiom 2 ($\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$) for *W* because it holds for all vectors in *V* and consequently for all vectors in *W*.
  - Other axioms inherited by *W* from *V* are 3, 7, 8, 9, and 10.
  - Only Axioms 1, 4, 5, 6 are needed to be checked.

Theorem 4.2.1

- Theorem 4.2.1
  - If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold:
    a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\underline{\mathbf{u} + \mathbf{v} \text{ is in } W}$.
    b) If k is any scalar and $\mathbf{u}$ is any vector in W, then $\underline{k\mathbf{u} \text{ is in } W}$.

Proof of Theorem 4.2.1

Axiom 1: If **u** and **v** are objects in *V*, then **u** + **v** is in *V*.

Axiom 6: If *k* is any scalar and **u** is any object in *V*, then *k***u** is in *V*.

- If *W* is a subspace of *V*, then all the vector space axioms are satisfied, including Axioms 1 and 6, which are precisely conditions (a) and (b).
- Conversely, assume conditions (a) and (b) hold. Since these conditions are vector space Axioms 1 and 6, we need only show that *W* satisfies the remaining eight axioms.

Proof of Theorem 4.2.1

- Axioms 2, 3, 7, 8, 9, and 10 are automatically satisfied by the vectors in W since they are satisfied by all vectors in V.
  - Therefore, we need only verify Axioms 4 and 5.

- Let $\mathbf{u}$ be any vector in W. By condition (b), $k\mathbf{u}$ is in W for every scalar k.
- Setting $k=0$, $0\mathbf{u} = \mathbf{0}$ is in W, and setting $k=-1$, $(-1)\mathbf{u} = -\mathbf{u}$ is in W – Axioms 4 and 5 hold in W

Remark

- Theorem 4.2.1 states that ***W*** is a subspace of ***V*** if and only if ***W*** is a closed under addition (condition (a)) and closed under scalar multiplication (condition (b)).

Example

- Let W be any plane through the origin and let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in W.
  - $\mathbf{u} + \mathbf{v}$ must lie in W since it is the diagonal of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, and $k\,\mathbf{u}$ must line in W for any scalar k since $k\,\mathbf{u}$ lies on a line through $\mathbf{u}$.
- Thus, W is closed under addition and scalar multiplication, so it is a subspace of R^3.

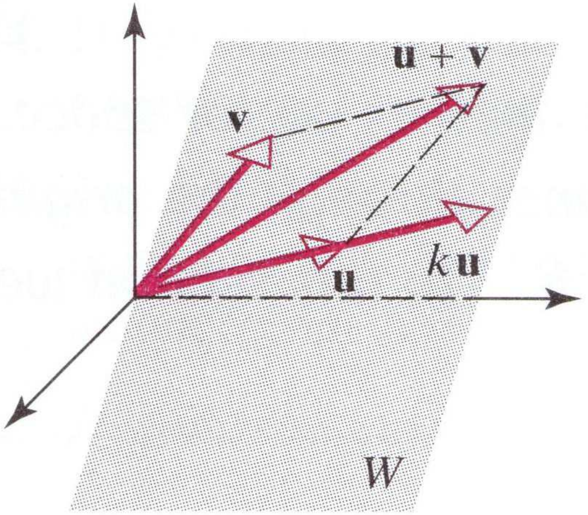
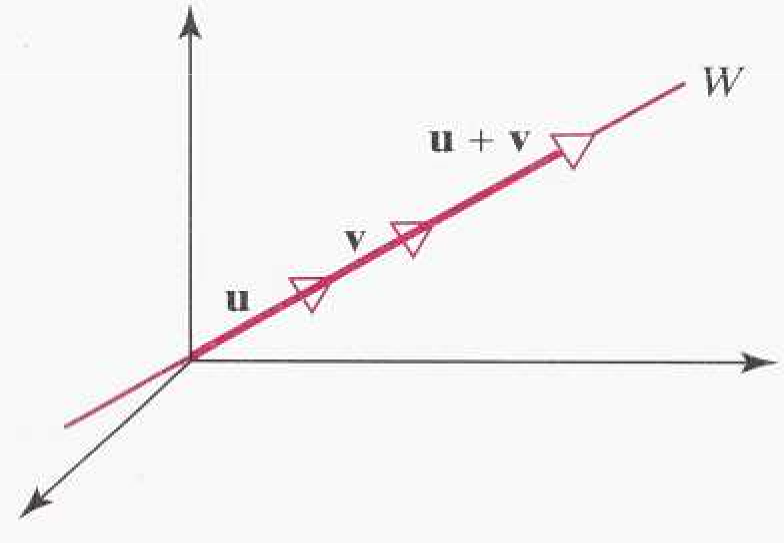


Figure 5.2.1
The vectors $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ both lie in the same plane as $\mathbf{u}$ and $\mathbf{v}$.

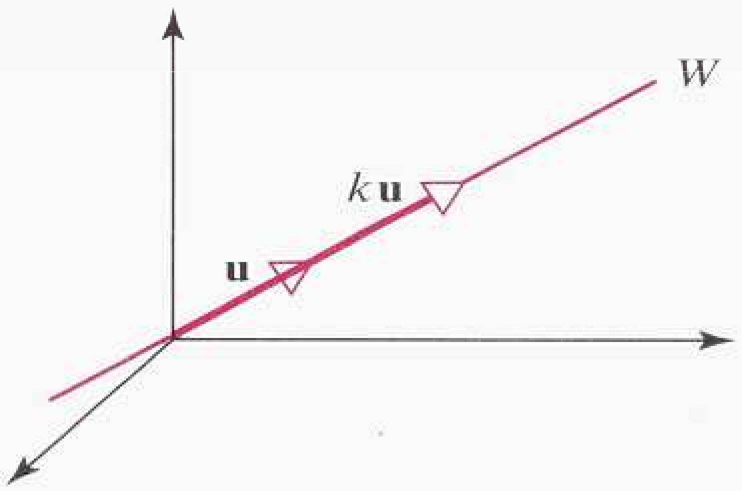
Example

- A line through the origin of R^3 is a subspace of R^3.

- Let W be a line through the origin of R^3.



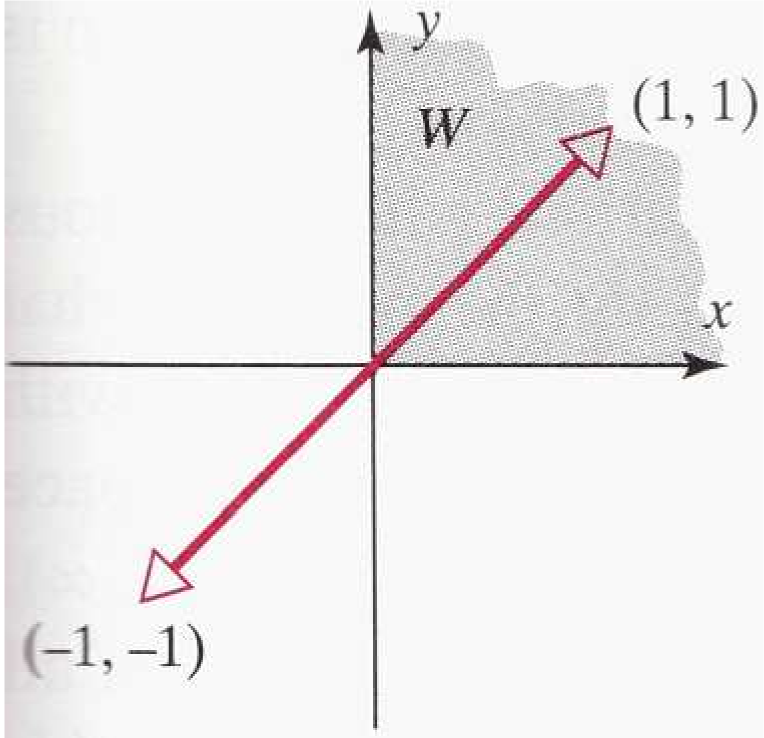
(a) W is closed under addition.



(b) W is closed under scalar multiplication.

Example (Not a Subspace)

- Let *W* be the set of all points (x, y) in R^2 such that $x \geq 0$ and $y \geq 0$. These are the points in the first quadrant.
- The set *W* is not a subspace of R^2 since it is not closed under scalar multiplication.
- For example, **v** = (1, 1) lines in *W*, but its negative (-1)**v** = -**v** = (-1, -1) does not.



Remarks

Think about “set” and “empty set”!

- Every nonzero vector space V has at least two subspace: V itself is a subspace, and the set $\{\mathbf{0}\}$ consisting of just the zero vector in V is a subspace called the zero subspace.
- Examples of subspaces of R^2 and R^3:
  - Subspaces of R^2:
    - $\{\mathbf{0}\}$
    - Lines through the origin
    - R^2
  - Subspaces of R^3:
    - $\{\mathbf{0}\}$
    - Lines through the origin
    - Planes through origin
    - R^3
- They are actually the only subspaces of R^2 and R^3

Subspaces of M_{nn}

- Since the sum of two symmetric matrices is symmetric, and a scalar multiple of a symmetric matrix is symmetric. Thus, the set of $n \times n$ symmetric matrices is a subspace of the vector space M_{nn} of $n \times n$ matrices.
- Similarly, the set of $n \times n$ upper triangular matrices, the set of $n \times n$ lower triangular matrices, and the set of $n \times n$ diagonal matrices all form subspaces of M_{nn}, since each of these sets is closed under addition and scalar multiplication.

A Subset of M_{nn} That is Not a Subspace

- The set W of invertible $n \times n$ matrices is not a subspace of M_{nn}.

$$U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \qquad V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$$

- The matrix $0U$ is the 2×2 zero matrix and hence is not invertible – ***not closure under scalar multiplication***.
- The matrix $U+V$ has a column of zeros, so it is not invertible – ***not closure under addition***.