Lecture 23: 6.2
Angle and Orthogonality

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2008/12/19

Theorem 6.2.1

- Theorem 6.2.1 (Cauchy-Schwarz Inequality)
  - If **u** and **v** are vectors in a real inner product space, then

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

  - The inequality can be written in the following two forms

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle^2 \leq \langle \boldsymbol{u}, \boldsymbol{u}\rangle\langle \boldsymbol{v}, \boldsymbol{v}\rangle$$

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle^2 \leq \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2$$

  - The Cauchy-Schwarz inequality for R^n (Theorem 4.1.3) follows as a special case of this theorem by taking $\langle \mathbf{u}, \mathbf{v}\rangle$ to be the Euclidean inner product $\mathbf{u} \cdot \mathbf{v}$.

Theorem 6.2.2

- Theorem 6.2.2 (Properties of Length)
  - If **u** and **v** are vectors in an inner product space V, and if k is any scalar, then :
    - $\|\mathbf{u}\| \geq 0$
    - $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
    - $\|k\mathbf{u}\| = |k|\,\|\mathbf{u}\|$
    - $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (Triangle inequality)
- Proof of (d)

$$
\begin{aligned}
&\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \langle \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}\rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{u}\rangle + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \langle \boldsymbol{v}, \boldsymbol{v}\rangle \\
&\leq \langle \boldsymbol{u}, \boldsymbol{u}\rangle + 2|\boldsymbol{u}, \boldsymbol{v}| + \langle \boldsymbol{v}, \boldsymbol{v}\rangle \qquad \text{(Property of absolute value)} \\
&\leq \langle \boldsymbol{u}, \boldsymbol{u}\rangle + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \langle \boldsymbol{v}, \boldsymbol{v}\rangle \qquad \text{(Theorem 6.2.1)} \\
&= \|\boldsymbol{u}\|^2 + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \|\boldsymbol{v}\|^2 \\
&= (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)^2
\end{aligned}
$$

Theorem 6.2.3

- Theorem 6.2.3 (Properties of Distance)
  - If **u**, **v**, and **w** are vectors in an inner product space V, and if k is any scalar, then:
    - $d(\mathbf{u}, \mathbf{v}) \geq 0$
    - $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
    - $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
    - $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (Triangle inequality)

Angle Between Vectors

- Cauchy-Schwarz inequality can be used to define angles in general inner product cases.

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle^2 \leq \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2$$

$$\Rightarrow \left[\frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| \|\boldsymbol{v}\|} \right]^2 \leq 1 \qquad \Rightarrow -1 \leq \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| \|\boldsymbol{v}\|} \leq 1$$

$$\cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\| \|\boldsymbol{v}\|} \qquad 0 \leq \theta \leq \pi$$

- We define θ to be the angle between **u** and **v**.

Example

- Let R^4 have the Euclidean inner product. Find the cosine of the angle θ between the vectors **u** = (4, 3, 1, -2) and **v** = (-2, 1, 2, 3).

$$\|\boldsymbol{u}\| = \sqrt{30} \qquad \|\boldsymbol{v}\| = \sqrt{18} \qquad \langle \boldsymbol{u}, \boldsymbol{v} \rangle = -9$$

$$\cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = -\frac{9}{\sqrt{30}\sqrt{18}} = -\frac{3}{2\sqrt{15}}$$

Orthogonality

- Definition
  - Two vectors **u** and **v** in an inner product space are called orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

- Example ($\langle U, V \rangle = \mathrm{tr}(U^T V) = \mathrm{tr}(V^T U) = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$)
  - If M_{22} has the inner project defined previously, then the matrices

$$U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

are orthogonal, since $\langle U, V \rangle = 1(0) + 0(2) + 1(0) + 1(0) = 0$.

Orthogonal Vectors in P_2

- Let P_2 have the inner product $<\mathbf{p},\ \mathbf{q}>=\int_{-1}^{1} p(x)q(x)dx$ and let $\mathbf{p} = x$ and $\mathbf{q} = x^2$.

- Then

$$\|\mathbf{p}\| =<\mathbf{p},\ \mathbf{p}>^{1/2}=\left[\int_{-1}^{1} xxdx\right]^{1/2} = \left[\int_{-1}^{1} x^2dx\right]^{1/2} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{q}\| =<\mathbf{q},\ \mathbf{q}>^{1/2}=\left[\int_{-1}^{1} x^2x^2dx\right]^{1/2} = \left[\int_{-1}^{1} x^4dx\right]^{1/2} = \sqrt{\frac{2}{5}}$$

$$<\mathbf{p},\mathbf{q}>=\int_{-1}^{1} xx^2dx = \int_{-1}^{1} x^3dx = 0$$

because $\langle\mathbf{p},\mathbf{q}\rangle = 0$, the vectors $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal relative to the given inner product.

Theorem 6.2.4 (Generalized Theorem of Pythagoras)

- If **u** and **v** are orthogonal vectors in an inner product space, then

$$\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2$$

Example

- Since $\mathbf{p} = x$ and $\mathbf{q} = x^2$ are orthogonal relative to the inner product $<\mathbf{p}, \mathbf{q}> = \int_{-1}^{1} p(x)q(x)dx$ on P_2.
- It follows from the Theorem of Pythagoras that

$$\|\mathbf{p} + \mathbf{q}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2$$

- Thus, from the previous example:

$$\|\mathbf{p}+\mathbf{q}\|^2 = (\sqrt{\frac{2}{3}})^2 + (\sqrt{\frac{2}{5}})^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}$$

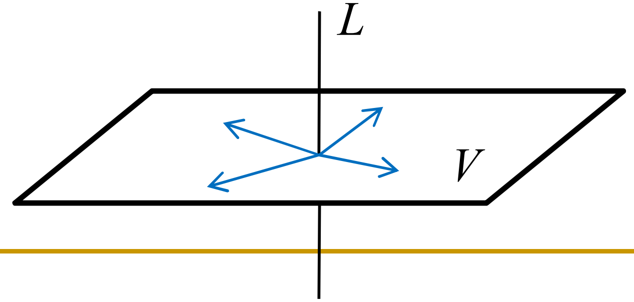
- We can check this result by direct integration:

$$\|\mathbf{p}+\mathbf{q}\|^2 = <\mathbf{p}+\mathbf{q}, \mathbf{p}+\mathbf{q}> = \int_{-1}^{1} (x+x^2)(x+x^2)dx$$

$$= \int_{-1}^{1} x^2 dx + 2\int_{-1}^{1} x^3 dx + \int_{-1}^{1} x^4 dx = \frac{2}{3} + 0 + \frac{2}{5} = \frac{16}{15}$$

Orthogonality

- Definition
  - Let W be a subspace of an inner product space V. A vector **u** in V is said to be orthogonal to W if it is orthogonal to every vector in W, and the set of all vectors in V that are orthogonal to W is called the orthogonal complement (正交補餘) of W.
  - If V is a plane through the origin of R^3 with Euclidean inner product, then the set of all vectors that are orthogonal to every vector in V forms the line L through the origin that is perpendicular to V.



Theorem 6.2.5

- Theorem 6.2.5 (Properties of Orthogonal Complements)
  - If W is a subspace of a finite-dimensional inner product space V, then:
    - $W^{\perp}$ is a subspace of V.
    - The only vector common to W and $W^{\perp}$ is $\mathbf{0}$; that is ,$W \cap W^{\perp} = \mathbf{0}$.
    - The orthogonal complement of $W^{\perp}$ is W; that is , $(W^{\perp})^{\perp} = W$.

Proof of Theorem 6.2.5(a)

- Note first that $\langle \mathbf{0}, \mathbf{w} \rangle = \mathbf{0}$ for every vector **w** in W, so $W^{\perp}$ contains at least the zero vector.
- We want to show that the sum of two vectors in $W^{\perp}$ is orthogonal to every vector in W (closed under addition) and that any scalar multiple of a vector in $W^{\perp}$ is orthogonal to every vector in W (closed under scalar multiplication).

Proof of Theorem 6.2.5(a)

- Let **u** and **v** be any vector in $W^{\perp}$, let k be any scalar, and let **w** be any vector in W. Then from the definition of $W^{\perp}$, we have $\langle \mathbf{u}, \mathbf{w} \rangle = 0$ and $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.
- Using the basic properties of the inner product, we have

$$\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle = 0 + 0 = 0$$

$$\langle k\mathbf{u}, \mathbf{w} \rangle = k\langle \mathbf{u}, \mathbf{w} \rangle = k(0) = 0$$

- Which proves that **u**+**v** and k**u** are in $W^{\perp}$.

Proof of Theorem 6.2.5(b)

The only vector common to W and $W^{\perp}$ is $\mathbf{0}$; that is ,$W \cap W^{\perp} = \mathbf{0}$.

- If $\mathbf{v}$ is common to W and $W^{\perp}$, then $\langle \mathbf{v}, \mathbf{v} \rangle=0$, which implies that $\mathbf{v}=\mathbf{0}$ by Axiom 4 for inner products.

Theorem 6.2.6

- Theorem 6.2.6
  - If A is an $m \times n$ matrix, then:
    - The nullspace of A and the row space of A are orthogonal complements in R^n with respect to the Euclidean inner product.
    - The nullspace of A^T and the column space of A are orthogonal complements in R^m with respect to the Euclidean inner product.

Proof of Theorem 6.2.6(a)

- We must show that if a vector **v** is orthogonal to every vector in the row space, then $A\mathbf{v}=\mathbf{0}$, and conversely, that if $A\mathbf{v}=\mathbf{0}$, then **v** is orthogonal to every vector in the row space.
- Assume that **v** is orthogonal to every vector in the row space of A. Then in particular, v is orthogonal to the row vectors $\mathbf{r}_1$, $\mathbf{r}_2$, …, $\mathbf{r}_m$ of A; that is

$$\boldsymbol{r}_1 \cdot \boldsymbol{v} = \boldsymbol{r}_2 \cdot \boldsymbol{v} = \cdots = \boldsymbol{r}_m \cdot \boldsymbol{v} = 0$$

Proof of Theorem 6.2.6(a)

$$\boldsymbol{r}_1 \cdot \boldsymbol{v} = \boldsymbol{r}_2 \cdot \boldsymbol{v} = \cdots = \boldsymbol{r}_m \cdot \boldsymbol{v} = 0$$

- The linear system $A\mathbf{x}=\mathbf{0}$ can be expressed in dot product notation as
$$\begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{x} \\ \boldsymbol{r}_2 \cdot \boldsymbol{x} \\ \vdots \\ \boldsymbol{r}_m \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
- And it follows that **v** is a solution of this system and hence lies in the nullspace of A.

Proof of Theorem 6.2.6(a)

- Conversely, assume that **v** is a vector in the nullspace of A, so A**v**=**0**. It follows that $\boldsymbol{r}_1 \cdot \boldsymbol{v} = \boldsymbol{r}_2 \cdot \boldsymbol{v} = \cdots = \boldsymbol{r}_m \cdot \boldsymbol{v} = 0$
- But if **r** is any vector in the row space of A, then **r** is expressible as a linear combination of the row vectors of A, say $\boldsymbol{r} = c_1\boldsymbol{r}_1 + c_2\boldsymbol{r}_2 + \cdots + c_m\boldsymbol{r}_m$
- Thus

$$\begin{aligned}\boldsymbol{r} \cdot \boldsymbol{v} &= (c_1\boldsymbol{r}_1 + c_2\boldsymbol{r}_2 + \cdots + c_m\boldsymbol{r}_m) \cdot \boldsymbol{v} \\ &= c_1(\boldsymbol{r}_1 \cdot \boldsymbol{v}) + c_2(\boldsymbol{r}_2 \cdot \boldsymbol{v}) + \cdots + c_m(\boldsymbol{r}_m \cdot \boldsymbol{v}) \\ &= 0 + 0 + \cdots + 0 = 0\end{aligned}$$

- Which proves that **v** is orthogonal to every vector in the row space of A.

Example (Basis for an Orthogonal Complement)

- Let W be the subspace of R^5 spanned by the vectors $\mathbf{w}_1$=(2, 2, -1, 0, 1), $\mathbf{w}_2$=(-1, -1, 2, -3, 1), $\mathbf{w}_3$=(1, 1, -2, 0, -1), $\mathbf{w}_4$=(0, 0, 1, 1, 1). Find a basis for the orthogonal complement of W.
- Solution
  - The space W spanned by $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ is the same as the row space of the matrix

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

  - By Theorem 6.2.6, the nullspace of A is the orthogonal complement of W.
  - In Example 4 of Section 5.5 we showed that

$$\mathbf{v_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

    form a basis for this nullspace.
  - Thus, vectors $\mathbf{v}_1$ = (-1, 1, 0, 0, 0) and $\mathbf{v}_2$ = (-1, 0, -1, 0, 1) form a basis for the orthogonal complement of W.

Remarks

- In any inner product space V, the zero space $\{\mathbf{0}\}$ and the entire space V are orthogonal complements.
- If A is an $n \times n$ matrix, to say that $A\mathbf{x}=\mathbf{0}$ has only the trivial solution is equivalent to saying that the orthogonal complement of the nullspace of A is all of R^n, or equivalently, that the row space of A is all of R^n.

Theorem 6.2.7 (Equivalent Statements)

- If A is an $m \times n$ matrix, and if $T_A : R^n \to R^n$ is multiplication by A, then the following are equivalent:
  - A is invertible.
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  - The reduced row-echelon form of A is I_n.
  - A is expressible as a product of elementary matrices.
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
  - $\det(A) \neq 0$.
  - The range of T_A is R^n.
  - T_A is one-to-one.
  - The column vectors of A are linearly independent.
  - The row vectors of A are linearly independent.
  - The column vectors of A span R^n.
  - The row vectors of A span R^n.
  - The column vectors of A form a basis for R^n.
  - The row vectors of A form a basis for R^n.
  - A has rank n.
  - A has nullity 0.
  - The orthogonal complement of the nullspace of A is R^n.
  - The orthogonal complement of the row of A is $\{\mathbf{0}\}$.

Lecture 23: 6.3 Orthonormal Bases

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2008/12/19

Orthonormal Basis

- Definition
  - A set of vectors in an inner product space is called an orthogonal set if all pairs of distinct vectors in the set are orthogonal.
  - An orthogonal set in which each vector has norm 1 is called orthonormal (單範正交).

- Example
  - Let $\mathbf{u}_1 = (0, 1, 0)$, $\mathbf{u}_2 = (1, 0, 1)$, $\mathbf{u}_3 = (1, 0, -1)$ and assume that R^3 has the Euclidean inner product.
  - It follows that the set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal since

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, \mathbf{u}_3 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle = 0.$$

Orthonormal

- If **v** is a nonzero vector in an inner product space, then the vector $\frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$ has norm 1
- Since $\left\|\frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}\right\| = \left|\frac{1}{\|\boldsymbol{v}\|}\right| \|\boldsymbol{v}\| = \frac{1}{\|\boldsymbol{v}\|}\|\boldsymbol{v}\| = 1$
- The process of multiplying a nonzero vector **v** by the reciprocal of its length to obtain a unit vector is called ***normalizing*** (正規化) **v**.
- An orthogonal set of nonzero vectors can always be converted to an orthonormal set by normalizing each of its vectors.

Example

- The Euclidean norms of the vectors are

$$\|\mathbf{u}_1\| = 1,\ \ \|\mathbf{u}_2\| = \sqrt{2},\ \ \|\mathbf{u}_3\| = \sqrt{2}$$

- Normalizing $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ yields

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = (0,1,0),\quad \mathbf{v}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = (\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}),\quad \mathbf{v}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = (\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}})$$

- The set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is <u>orthonormal</u> since

$$\langle \mathbf{v}_1, \mathbf{v}_2\rangle = \langle \mathbf{v}_1, \mathbf{v}_3\rangle = \langle \mathbf{v}_2, \mathbf{v}_3\rangle = 0 \quad \textit{and} \quad \|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \|\mathbf{v}_3\| = 1$$

Orthonormal Basis

- In an inner product space, a basis consisting of orthonormal vectors is called an orthonormal basis, and a basis consisting of orthogonal vectors is called an orthogonal basis.
- A familiar example of an orthornormal basis is the standard basis for R^3

$$\mathbf{i}=(1,0,0),\ \mathbf{j}=(0,1,0),\ \mathbf{k}=(0,0,1)$$

- The standard basis for Rn

$$\mathbf{e}_1=(1,0,0,\ldots,0),\ \mathbf{e}_2=(0,1,0,\ldots,0),\ \ldots,\ \mathbf{e}_n=(0,0,0,\ldots,1)$$

Orthonormal Basis

- Theorem 6.3.1*
  - If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$$

- Remark
  - The scalars $\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{u}, \mathbf{v}_n \rangle$ are the coordinates of the vector $\mathbf{u}$ relative to the orthonormal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and

$$(\mathbf{u})_S = (\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{u}, \mathbf{v}_n \rangle)$$

    is the coordinate vector of $\mathbf{u}$ relative to this basis

Proof of Theorem 6.3.1

- Since $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ is a basis, a vector $\mathbf{u}$ can be expressed in the form $\mathbf{u}=k_1\mathbf{v}_1+k_2\mathbf{v}_2+\ldots+k_n\mathbf{v}_n$
- We shall show that $k_i=\langle\mathbf{u},\mathbf{v}_i\rangle$ for $i=1, 2,\ldots, n$. For each vector $\mathbf{v}_i$ in S, we have

$$\langle\mathbf{u},\mathbf{v}_i\rangle = \langle k_1\mathbf{v}_1+k_2\mathbf{v}_2+\ldots+k_n\mathbf{v}_n, \mathbf{v}_i\rangle$$
$$= k_1\langle\mathbf{v}_1,\mathbf{v}_i\rangle + k_2\langle\mathbf{v}_2,\mathbf{v}_i\rangle+\ldots+ k_n\langle\mathbf{v}_n,\mathbf{v}_i\rangle$$

- Since S is an orthonormal set, we have

$$\langle \boldsymbol{v}_i, \boldsymbol{v}_i\rangle = \|\boldsymbol{v}_i\|^2 = 1 \qquad \langle \boldsymbol{v}_j, \boldsymbol{v}_i\rangle = 0 \quad \text{if } j \neq i$$

- Therefore, $\langle\mathbf{u},\mathbf{v}_i\rangle = k_i$

Example

- Let $\mathbf{v}_1 = (0, 1, 0)$, $\mathbf{v}_2 = (-4/5, 0, 3/5)$, $\mathbf{v}_3 = (3/5, 0, 4/5)$.
  It is easy to check that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis for R^3 with the Euclidean inner product.
  Express the vector $\mathbf{u} = (1, 1, 1)$ as a linear combination of the vectors in S, and find the coordinate vector $(\mathbf{u})_s$.

- Solution:
  - $\langle \mathbf{u}, \mathbf{v}_1 \rangle = 1$, $\langle \mathbf{u}, \mathbf{v}_2 \rangle = -1/5$, $\langle \mathbf{u}, \mathbf{v}_3 \rangle = 7/5$
  - Therefore, by Theorem 6.3.1 we have $\mathbf{u} = \mathbf{v}_1 - 1/5\ \mathbf{v}_2 + 7/5\ \mathbf{v}_3$
  - That is, $(1, 1, 1) = (0, 1, 0) - 1/5\ (-4/5, 0, 3/5) + 7/5\ (3/5, 0, 4/5)$
  - The coordinate vector of $\mathbf{u}$ relative to S is

$$(\mathbf{u})_s=(\langle \mathbf{u}, \mathbf{v}_1 \rangle, \langle \mathbf{u}, \mathbf{v}_2 \rangle, \langle \mathbf{u}, \mathbf{v}_3 \rangle) = (1, -1/5, 7/5)$$

Theorem 6.3.2

- Theorem 6.3.2
  - If S is an orthonormal basis for an n-dimensional inner product space, and if $(\mathbf{u})_s = (u_1, u_2, \ldots, u_n)$ and $(\mathbf{v})_s = (v_1, v_2, \ldots, v_n)$ then:

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- Remark
  - By working with orthonormal bases, the computation of general norms and inner products can be reduced to the computation of Euclidean norms and inner products of the coordinate vectors.

Example

- If R^3 has the Euclidean inner product, then the norm of the vector **u**=(1,1,1) is

$$\|\boldsymbol{u}\| = (\boldsymbol{u} \cdot \boldsymbol{u})^{1/2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

- However, if we let R^3 have the orthonormal basis S in the last example, then we know from that the coordinate vector of **u** relative to S is $(\mathbf{u})_s$= (1, -1/5, 7/5)
- The norm of **u** yields

$$\|\boldsymbol{u}\| = \sqrt{1^2 + (-\tfrac{1}{5})^2 + (\tfrac{7}{5})^2} = \sqrt{\tfrac{75}{25}} = \sqrt{3}$$

Coordinates Relative to Orthogonal Bases

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for a vector space V, then normalizing each of these vectors yields the orthonormal basis

$$S' = \left\{ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \cdots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|} \right\}$$

- Thus, if **u** is any vector in V, it follows from theorem 6.3.1 that

$$\mathbf{u} = \left\langle \mathbf{u}, \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \right\rangle \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} + \left\langle \mathbf{u}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \right\rangle \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} + \cdots + \left\langle \mathbf{u}, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|} \right\rangle \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$$

or

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2} \mathbf{v}_n$$

- The above equation expresses **u** as a linear combination of the vectors in the orthogonal basis S.

Theorem 6.3.3

- Theorem 6.3.3
  - If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

Proof of Theorem 6.3.3

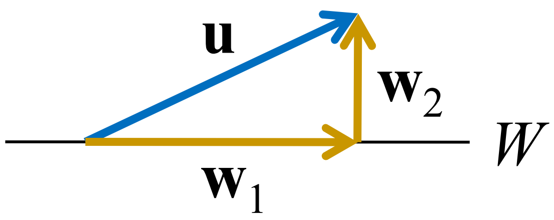
- Assume that $k_1\mathbf{v}_1+k_2\mathbf{v}_2+\ldots+k_n\mathbf{v}_n = \mathbf{0}$. To demonstrate that S is linearly independent, we must prove that $k_1=k_2=\ldots=0$.
- For each $\mathbf{v}_i$ in S, $\langle k_1\mathbf{v}_1+k_2\mathbf{v}_2+\ldots+k_n\mathbf{v}_n, \mathbf{v}_i\rangle = \langle \mathbf{0},\mathbf{v}_i\rangle=0$
  or, equivalently $k_1\langle \mathbf{v}_1,\mathbf{v}_i\rangle+ k_2\langle \mathbf{v}_2,\mathbf{v}_i\rangle+\ldots+ k_n\langle \mathbf{v}_n,\mathbf{v}_i\rangle=0$
- From the orthogonality of S it follows that $\langle \mathbf{v}_j,\mathbf{v}_i\rangle=0$ when j is not equal to i, so the equation reduces to $k_i\langle \mathbf{v}_i,\mathbf{v}_i\rangle =0$
- Since the vectors in S are assumed to be nonzero, $\langle \mathbf{v}_i,\mathbf{v}_i\rangle \neq 0$. Therefore, $k_i=0$. Since the subscript i is arbitrary, we have $k_1=k_2=\ldots=k_n=0$.

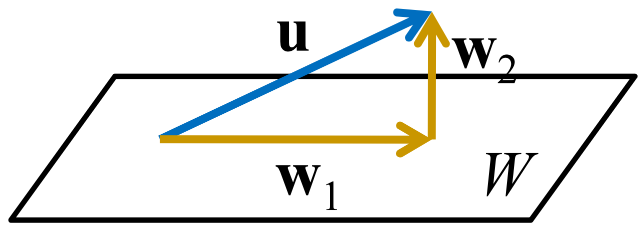
Theorem 6.3.4

- Theorem 6.3.4 (Projection Theorem)
  - If W is a finite-dimensional subspace of an product space V, then every vector $\mathbf{u}$ in V can be expressed in exactly one way as

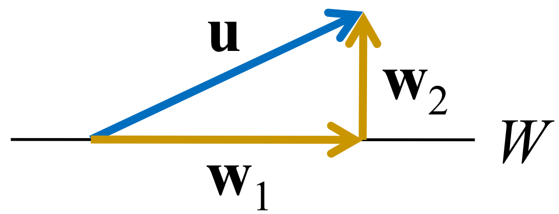
$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^{\perp}$.

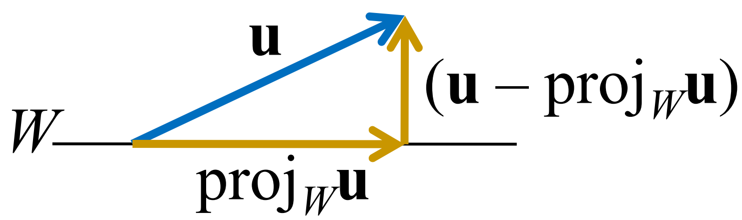




Projection



- The vector $\mathbf{w}_1$ is called the orthogonal projection of $\mathbf{u}$ on W and is denoted $\text{proj}_W\mathbf{u}$.
- The vector $\mathbf{w}_2$ is called the component of $\mathbf{u}$ orthogonal to W and is denote by $\text{proj}_{W\perp}\,\mathbf{u}$.
- $\mathbf{u} = \text{proj}_W\mathbf{u} + \text{proj}_{W\perp}\,\mathbf{u}$
- Since $\mathbf{w}_2 = \mathbf{u}\text{-}\mathbf{w}_1$, it follows that $\text{proj}_{W\perp}\,\mathbf{u} = \mathbf{u} - \text{proj}_W\mathbf{u}$
- So we can write $\mathbf{u} = \text{proj}_W\mathbf{u} + (\mathbf{u} - \text{proj}_W\mathbf{u})$



Theorem 6.3.5

- Theorem 6.3.5
  - Let W be a finite-dimensional <u>subspace</u> of an inner product space V.
    - If $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an <u>orthonormal basis</u> for W, and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r$$

    - If $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an <u>orthogonal basis</u> for W, and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2} \mathbf{v}_r$$

⇦ Need Normalization

Example

- Let R^3 have the Euclidean inner product, and let W be the subspace spanned by the orthonormal vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-4/5, 0, 3/5)$.
- From the above theorem, the orthogonal projection of $\mathbf{u} = (1, 1, 1)$ on W is $\text{proj}_W \mathbf{u} = <\mathbf{u}, \mathbf{v}_1> \mathbf{v}_1 + <\mathbf{u}, \mathbf{v}_2> \mathbf{v}_2$

$$= (1)(0,\ 1,\ 0) + (-\frac{1}{5})(-\frac{4}{5},\ 0,\ \frac{3}{5}) = (\frac{4}{25},\ 1,\ -\frac{3}{25})$$

- The component of $\mathbf{u}$ orthogonal to W is

$$\text{proj}_{W^\perp} \mathbf{u} = \mathbf{u} - \text{proj}_W \mathbf{u} = (1,\ 1,\ 1) - (\frac{4}{25},\ 1,\ -\frac{3}{25}) = (\frac{21}{25},\ 0,\ \frac{28}{25})$$

- Observe that $\text{proj}_{W^\perp}\mathbf{u}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$.