Lecture 23: 6.1
Inner Products

Wei-Ta Chu

2008/12/17
Definition

- An **inner product** on a real vector space $V$ is a function that associates a real number $\langle u, v \rangle$ with each pair of vectors $u$ and $v$ in $V$ in such a way that the following axioms are satisfied for all vectors $u$, $v$, and $w$ in $V$ and all scalars $k$.
  - $\langle u, v \rangle = \langle v, u \rangle$
  - $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
  - $\langle ku, v \rangle = k \langle u, v \rangle$
  - $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$

A real vector space with an **inner product** is called a **real inner product space**.
Euclidean Inner Product on $\mathbb{R}^n$

- If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbb{R}^n$, then the formula
  $$\langle \mathbf{v}, \mathbf{u} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$
defines $\langle \mathbf{v}, \mathbf{u} \rangle$ to be the Euclidean product on $\mathbb{R}^n$.

- The four inner product axioms hold by Theorem 4.1.2.
Properties of Euclidean Inner Product

Theorem 4.1.2

If \( \mathbf{u} \), \( \mathbf{v} \) and \( \mathbf{w} \) are vectors in \( \mathbb{R}^n \) and \( k \) is any scalar, then

- \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \)
- \( (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \)
- \( (k \mathbf{u}) \cdot \mathbf{v} = k (\mathbf{u} \cdot \mathbf{v}) \)
- \( \mathbf{v} \cdot \mathbf{v} \geq 0 \); Further, \( \mathbf{v} \cdot \mathbf{v} = 0 \) if and only if \( \mathbf{v} = \mathbf{0} \)

Example

- \( (3\mathbf{u} + 2\mathbf{v}) \cdot (4\mathbf{u} + \mathbf{v}) \)
  \[ = (3\mathbf{u}) \cdot (4\mathbf{u} + \mathbf{v}) + (2\mathbf{v}) \cdot (4\mathbf{u} + \mathbf{v}) \]
  \[ = (3\mathbf{u}) \cdot (4\mathbf{u}) + (3\mathbf{u}) \cdot \mathbf{v} + (2\mathbf{v}) \cdot (4\mathbf{u}) + (2\mathbf{v}) \cdot \mathbf{v} \]
  \[ = 12(\mathbf{u} \cdot \mathbf{u}) + 11(\mathbf{u} \cdot \mathbf{v}) + 2(\mathbf{v} \cdot \mathbf{v}) \]
Euclidean Inner Product vs. Inner Product

- The Euclidean inner product is the most important inner product on $\mathbb{R}^n$. However, there are various applications in which it is desirable to modify the Euclidean inner product by weighting its terms differently.

- More precisely, if $w_1, w_2, \ldots, w_n$ are positive real numbers, and if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbb{R}^n$, then it can be shown

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

which is called *weighted Euclidean inner product with weights* $w_1, w_2, \ldots, w_n \ldots$
Example

- Suppose that some physical experiment can produce any of $n$ possible numerical values $x_1, x_2, \ldots, x_n$.
- We perform $m$ repetitions of the experiment and yield these values with various frequencies; that is, $x_1$ occurs $f_1$ times, $x_2$ occurs $f_2$ times, and so forth. $f_1 + f_2 + \ldots + f_n = m$.
- Thus, the **arithmetic average**, or **mean**, is

$$
\bar{x} = \frac{f_1 x_1 + f_2 x_2 + \cdots + f_n x_n}{f_1 + f_2 + \cdots + f_n} = \frac{1}{m} (f_1 x_1 + f_2 x_2 + \cdots + f_n x_n)
$$

Example

\[ \bar{x} = \frac{f_1 x_1 + f_2 x_2 + \cdots + f_n x_n}{f_1 + f_2 + \cdots + f_n} = \frac{1}{m} (f_1 x_1 + f_2 x_2 + \cdots + f_n x_n) \]

- If we let \( f=(f_1,f_2,\ldots,f_n), x=(x_1,x_2,\ldots,x_n), w_1=w_2=\ldots=w_n=1/m \)
- Then this equation can be expressed as the weighted inner product

\[ \bar{x} = \langle f, x \rangle = w_1 f_1 x_1 + w_2 f_2 x_2 + \cdots + w_n f_n x_n \]
Weighted Euclidean Product

- Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in $\mathbb{R}^2$. Verify that the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$ satisfies the four product axioms.

Solution:
- Note first that if $\mathbf{u}$ and $\mathbf{v}$ are interchanged in this equation, the right side remains the same. Therefore, $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.
- If $\mathbf{w} = (w_1, w_2)$, then
  
  $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = 3(u_1+v_1)w_1 + 2(u_2+v_2)w_2 = (3u_1w_1 + 2u_2w_2) + (3v_1w_1 + 2v_2w_2) = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

which establishes the second axiom.
Weighted Euclidean Product

- \( \langle ku, v \rangle = 3(ku_1)v_1 + 2(ku_2)v_2 = k(3u_1v_1 + 2u_2v_2) = k \langle u, v \rangle \)
  which establishes the third axiom.

- \( \langle v, v \rangle = 3v_1^2 + 2v_2^2 \geq 0 \). Obviously, \( \langle v, v \rangle = 3v_1^2 + 2v_2^2 = 0 \) if and only if \( v_1 = v_2 = 0 \), That is, if and only if \( v = (v_1, v_2) = 0 \). Thus, the fourth axiom is satisfied.
Definition

- If \( V \) is an inner product space, then the norm (or length) of a vector \( u \) in \( V \) is denoted by \( ||u|| \) and is defined by
  \[
  ||u|| = \left( \langle u, u \rangle \right)^{\frac{1}{2}}
  \]

- The distance between two points (vectors) \( u \) and \( v \) is denoted by \( d(u,v) \) and is defined by
  \[
  d(u, v) = ||u − v||
  \]

- If a vector has norm 1, then we say that it is a **unit vector**.
Norm and Distance in $\mathbb{R}^n$

- If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbb{R}^n$ with the Euclidean inner product, then

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = (\mathbf{u} \cdot \mathbf{u})^{1/2} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle^{1/2} = \left[ (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \right]^{1/2}$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$
Weighted Euclidean Inner Product

- The norm and distance depend on the inner product used.
  - If the inner product is changed, then the norms and distances between vectors also change.
  - For example, for the vectors $\mathbf{u} = (1,0)$ and $\mathbf{v} = (0,1)$ in $\mathbb{R}^2$ with the Euclidean inner product, we have
    \[
    \|\mathbf{u}\| = 1 \quad \text{and} \quad d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(1,-1)\| = \sqrt{1^2 - (-1)^2} = \sqrt{2}
    \]
  - However, if we change to the weighted Euclidean inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1 v_1 + 2u_2 v_2$, then we obtain
    \[
    \|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = (3 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 0)^{1/2} = \sqrt{3}
    \]
    \[
    d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \langle (1,-1), (1,-1) \rangle^{1/2} = [3 \cdot 1 \cdot 1 + 2 \cdot (-1) \cdot (-1)]^{1/2} = \sqrt{5}
    \]
Unit Circles and Spheres in Inner Product Space

- If $V$ is an inner product space, then the set of points in $V$ that satisfy

$$||u|| = 1$$

is called the unite sphere or sometimes the unit circle in $V$. In $R^2$ and $R^3$ these are the points that lie 1 unit away from the origin.
Unit Circles in $R^2$

- Sketch the unit circle in an $xy$-coordinate system in $R^2$ using the Euclidean inner product $\langle u, v \rangle = u_1v_1 + u_2v_2$
- Sketch the unit circle in an $xy$-coordinate system in $R^2$ using the Euclidean inner product $\langle u, v \rangle = \frac{1}{9}u_1v_1 + \frac{1}{4}u_2v_2$
- Solution
  - If $u = (x, y)$, then $||u|| = \langle u, u \rangle^{\frac{1}{2}} = (x^2 + y^2)^{\frac{1}{2}}$, so the equation of the unit circle is $x^2 + y^2 = 1$.
  - If $u = (x, y)$, then $||u|| = \langle u, u \rangle^{\frac{1}{2}} = (\frac{1}{9}x^2 + \frac{1}{4}y^2)^{\frac{1}{2}}$, so the equation of the unit circle is $x^2/9 + y^2/4 = 1$. 

![Diagram of unit circle in $R^2$]
Inner Products Generated by Matrices

Let \( u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \) and \( v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \) be vectors in \( \mathbb{R}^n \) (expressed as \( n \times 1 \) matrices), and let \( A \) be an invertible \( n \times n \) matrix.

If \( u \cdot v \) is the Euclidean inner product on \( \mathbb{R}^n \), then the formula

\[
\langle u, v \rangle = A u \cdot A v
\]

defines an inner product; it is called the inner product on \( \mathbb{R}^n \) generated by \( A \).

Recalling that the Euclidean inner product \( u \cdot v \) can be written as the matrix product \( v^T u \), the above formula can be written in the alternative form

\[
\langle u, v \rangle = (Av)^T A u,
\]
or equivalently,

\[
\langle u, v \rangle = v^T A^T A u
\]
Inner Product Generated by the Identity Matrix

- The inner product on $\mathbb{R}^n$ generated by the $n \times n$ identity matrix is the Euclidean inner product: Let $A = I$, we have $\langle u, v \rangle = Iu \cdot Iv = u \cdot v$

- The weighted Euclidean inner product $\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$ is the inner product on $\mathbb{R}^2$ generated by

\[
A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}
\]

since

\[
\langle u, v \rangle = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix} u_1 = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 3u_1v_1 + 2u_2v_2
\]

\[
\langle u, v \rangle = v^T A^T A u
\]
Inner Product Generated by the Identity Matrix

In general, the weighted Euclidean inner product \( \langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n \) is the inner product on \( \mathbb{R}^n \) generated by

\[
A = \begin{bmatrix}
\sqrt{w_1} & 0 & 0 & \cdots & 0 \\
0 & \sqrt{w_2} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \sqrt{w_n}
\end{bmatrix}
\]
An Inner Product on $M_{22}$

- If $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ are any two $2 \times 2$ matrices, then
  \[ \langle U, V \rangle = \text{tr}(U^T V) = \text{tr}(V^T U) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 \]
defines an inner product on $M_{22}$

- For example, if
  \[
  U = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}
  \]
then $\langle U, V \rangle = 1(-1) + 2(0) + 3(3) + 4(2) = 16$

- The norm of a matrix $U$ relative to this inner product is
  \[ \|U\| = \sqrt{\langle U, U \rangle} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2} \]
and the unit sphere in this space consists of all $2 \times 2$ matrices $U$ whose entries satisfy the equation $\|U\| = 1$, which on squaring yields $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$
An Inner Product on $P_2$

- If $p = a_0 + a_1 x + a_2 x^2$ and $q = b_0 + b_1 x + b_2 x^2$ are any two vectors in $P_2$, then the following formula defines an inner product on $P_2$:

$$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$$

- The norm of the polynomial $p$ relative to this inner product is

$$\|p\| = \langle p, p \rangle^{1/2} = \sqrt{a_0^2 + a_1^2 + a_2^2}$$

and the unit sphere in this space consists of all polynomials $p$ in $P_2$ whose coefficients satisfy the equation $\|p\| = 1$, which on squaring yields

$$a_0^2 + a_1^2 + a_2^2 = 1$$
Theorem 6.1.1 (Properties of Inner Products)

- If \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) are vectors in a real inner product space, and \( k \) is any scalar, then:
  - \( \langle 0, \mathbf{v} \rangle = \langle \mathbf{v}, 0 \rangle = 0 \)
  - \( \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle \)
  - \( \langle \mathbf{u}, k\mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle \)
  - \( \langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle \)
  - \( \langle \mathbf{u}, \mathbf{v} - \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{w} \rangle \)
Example

\[ \langle u - 2v, 3u + 4v \rangle \]
\[ = \langle u, 3u + 4v \rangle - \langle 2v, 3u + 4v \rangle \]
\[ = \langle u, 3u \rangle + \langle u, 4v \rangle - \langle 2v, 3u \rangle - \langle 2v, 4v \rangle \]
\[ = 3 \langle u, u \rangle + 4 \langle u, v \rangle - 6 \langle v, u \rangle - 8 \langle v, v \rangle \]
\[ = 3 \| u \|^2 + 4 \langle u, v \rangle - 6 \langle u, v \rangle - 8 \| v \|^2 \]
\[ = 3 \| u \|^2 - 2 \langle u, v \rangle - 8 \| v \|^2 \]
Example

- We are guaranteed without any further proof that the five properties given in Theorem 6.1.1 are true for the inner product on $\mathbb{R}^n$ generated by any matrix $A$.

\[ \langle u, v + w \rangle = (v+w)^T A^T A u \]
\[ = (v^T + w^T) A^T A u \quad \text{[Property of transpose]} \]
\[ = (v^T A^T A u) + (w^T A^T A u) \quad \text{[Property of matrix multiplication]} \]
\[ = \langle u, v \rangle + \langle u, w \rangle \]
Lecture 23: 6.2
Angle and Orthogonality

Wei-Ta Chu

2008/12/17
Theorem 6.2.1 (Cauchy-Schwarz Inequality)

If \( \mathbf{u} \) and \( \mathbf{v} \) are vectors in a real inner product space, then

\[
|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}|| \, ||\mathbf{v}||
\]

The inequality can be written in the following two forms

\[
\langle \mathbf{u}, \mathbf{v} \rangle^2 \leq \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle
\]

\[
\langle \mathbf{u}, \mathbf{v} \rangle^2 \leq ||\mathbf{u}||^2 ||\mathbf{v}||^2
\]

The Cauchy-Schwarz inequality for \( \mathbb{R}^n \) (Theorem 4.1.3) follows as a special case of this theorem by taking \( \langle \mathbf{u}, \mathbf{v} \rangle \) to be the Euclidean inner product \( \mathbf{u} \cdot \mathbf{v} \).
Theorem 6.2.2

Theorem 6.2.2 (Properties of Length)

- If \( u \) and \( v \) are vectors in an inner product space \( V \), and if \( k \) is any scalar, then:
  - \( ||u|| \geq 0 \)
  - \( ||u|| = 0 \) if and only if \( u = 0 \)
  - \( ||ku|| = |k||u|| \)
  - \( ||u + v|| \leq ||u|| + ||v|| \) (Triangle inequality)

Proof of (d)

\[
\begin{align*}
||u + v||^2 &= \langle u + v, u + v \rangle \\
&= \langle u, u \rangle + 2\langle u, v \rangle + \langle v, v \rangle \\
&\leq \langle u, u \rangle + 2||u||||v|| + \langle v, v \rangle \quad \text{(Property of absolute value)} \\
&\leq \langle u, u \rangle + 2||u||||v|| + \langle v, v \rangle \quad \text{(Theorem 6.2.1)} \\
&= ||u||^2 + 2||u||||v|| + ||v||^2 \\
&= (||u|| + ||v||)^2
\end{align*}
\]
Theorem 6.2.3 (Properties of Distance)

- If \( u, v, \) and \( w \) are vectors in an inner product space \( V \), and if \( k \) is any scalar, then:
  - \( d(u, v) \geq 0 \)
  - \( d(u, v) = 0 \) if and only if \( u = v \)
  - \( d(u, v) = d(v, u) \)
  - \( d(u, v) \leq d(u, w) + d(w, v) \) (Triangle inequality)
Angle Between Vectors

- Cauchy-Schwarz inequality can be used to define angles in general inner product cases.

\[
\langle u, v \rangle^2 \leq \|u\|^2 \|v\|^2
\]

\[
\left[ \frac{\langle u, v \rangle}{\|u\| \|v\|} \right]^2 \leq 1 \quad -1 \leq \frac{\langle u, v \rangle}{\|u\| \|v\|} \leq 1
\]

\[
\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|} \quad 0 \leq \theta \leq \pi
\]

- We define \( \theta \) to be the angle between \( u \) and \( v \).
Example

Let $\mathbb{R}^4$ have the Euclidean inner product. Find the cosine of the angle $\theta$ between the vectors $\mathbf{u} = (4, 3, 1, -2)$ and $\mathbf{v} = (-2, 1, 2, 3)$.

\[ \|\mathbf{u}\| = \sqrt{30} \quad \|\mathbf{v}\| = \sqrt{18} \quad \langle \mathbf{u}, \mathbf{v} \rangle = -9 \]

\[ \cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} = -\frac{9}{\sqrt{30} \sqrt{18}} = -\frac{3}{2 \sqrt{15}} \]
Orthogonality

- **Definition**
  - Two vectors $u$ and $v$ in an inner product space are called *orthogonal* if $\langle u, v \rangle = 0$.

- **Example** ($\langle U, V \rangle = \text{tr}(U^T V) = \text{tr}(V^T U) = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$)
  - If $M_{22}$ has the inner product defined previously, then the matrices
    
    $U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

    are orthogonal, since $\langle U, V \rangle = 1(0) + 0(2) + 1(0) + 1(0) = 0$. 
Orthogonal Vectors in $P_2$

- Let $P_2$ have the inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$ and let $p = x$ and $q = x^2$.

- Then

$$\|p\| = \langle p, p \rangle^{1/2} = \left[ \int_{-1}^{1} xx\,dx \right]^{1/2} = \left[ \int_{-1}^{1} x^2\,dx \right]^{1/2} = \sqrt{\frac{2}{3}}$$

$$\|q\| = \langle q, q \rangle^{1/2} = \left[ \int_{-1}^{1} x^2 x^2\,dx \right]^{1/2} = \left[ \int_{-1}^{1} x^4\,dx \right]^{1/2} = \sqrt{\frac{2}{5}}$$

$$\langle p, q \rangle = \int_{-1}^{1} xx^2\,dx = \int_{-1}^{1} x^3\,dx = 0$$

because $\langle p, q \rangle = 0$, the vectors $p = x$ and $q = x^2$ are orthogonal relative to the given inner product.
Theorem 6.2.4 (Generalized Theorem of Pythagoras)

- If \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal vectors in an inner product space, then

\[
\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2
\]

- Proof

\[
\| \mathbf{u} + \mathbf{v} \|^2 = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\
= \langle \mathbf{u}, \mathbf{u} \rangle + 2 \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\
= \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2
\]
Example

- Since \( p = x \) and \( q = x^2 \) are orthogonal relative to the inner product \( \langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx \) on \( P_2 \).
- It follows from the Theorem of Pythagoras that
  \[ \| p + q \|^2 = \| p \|^2 + \| q \|^2 \]
- Thus, from the previous example:
  \[ \| p+q \|^2 = (\sqrt{\frac{2}{3}})^2 + (\sqrt{\frac{2}{5}})^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{15} \]
- We can check this result by direct integration:
  \[ \| p+q \|^2 = \langle p+q, p+q \rangle = \int_{-1}^{1} (x + x^2)(x + x^2)dx \]
  \[ = \int_{-1}^{1} x^2 dx + 2 \int_{-1}^{1} x^3 dx + \int_{-1}^{1} x^4 dx = \frac{2}{3} + 0 + \frac{2}{5} = \frac{16}{15} \]
Orthogonality

- **Definition**
  - Let $W$ be a subspace of an inner product space $V$. A vector $u$ in $V$ is said to be **orthogonal to** $W$ if it is orthogonal to every vector in $W$, and the set of all vectors in $V$ that are orthogonal to $W$ is called the **orthogonal complement** of $W$.
  
  - If $V$ is a plane through the origin of $\mathbb{R}^3$ with Euclidean inner product, then the set of all vectors that are orthogonal to every vector in $V$ forms the line $L$ through the origin that is perpendicular to $V$. 

![Diagram](https://example.com/diagram.png)
Theorem 6.2.5 (Properties of Orthogonal Complements)

- If $W$ is a subspace of a finite-dimensional inner product space $V$, then:
  - $W^\perp$ is a subspace of $V$. (read “$W$ perp”)
  - The only vector common to $W$ and $W^\perp$ is $0$; that is, $W \cap W^\perp = \{0\}$.
  - The orthogonal complement of $W^\perp$ is $W$; that is, $(W^\perp)^\perp = W$. 
Proof of Theorem 6.2.5(a)

- Note first that $\langle 0, w \rangle = 0$ for every vector $w$ in $W$, so $W^\perp$ contains at least the zero vector.

- We want to show that the sum of two vectors in $W^\perp$ is orthogonal to every vector in $W$ (closed under addition) and that any scalar multiple of a vector in $W^\perp$ is orthogonal to every vector in $W$ (closed under scalar multiplication).
Proof of Theorem 6.2.5(a)

- Let $u$ and $v$ be any vector in $W^\perp$, let $k$ be any scalar, and let $w$ be any vector in $W$. Then from the definition of $W^\perp$, we have $\langle u, w \rangle = 0$ and $\langle v, w \rangle = 0$.
- Using the basic properties of the inner product, we have
  \[
  \langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle = 0 + 0 = 0
  \]
  \[
  \langle ku, w \rangle = k \langle u, w \rangle = k(0) = 0
  \]
- Which proves that $u+v$ and $ku$ are in $W^\perp$. 
Proof of Theorem 6.2.5(b)

The only vector common to $W$ and $W^\perp$ is $0$; that is, $W \cap W^\perp = 0$.

- If $v$ is common to $W$ and $W^\perp$, then $\langle v, v \rangle = 0$, which implies that $v = 0$ by Axiom 4 for inner products.
Theorem 6.2.6

- If $A$ is an $m \times n$ matrix, then:
  - The nullspace of $A$ and the row space of $A$ are orthogonal complements in $\mathbb{R}^n$ with respect to the Euclidean inner product.
  - The nullspace of $A^T$ and the column space of $A$ are orthogonal complements in $\mathbb{R}^m$ with respect to the Euclidean inner product.
Proof of Theorem 6.2.6(a)

- We must show that if a vector \( \mathbf{v} \) is orthogonal to every vector in the row space, then \( A\mathbf{v} = \mathbf{0} \), and conversely, that if \( A\mathbf{v} = \mathbf{0} \), then \( \mathbf{v} \) is orthogonal to every vector in the row space.

- Assume that \( \mathbf{v} \) is orthogonal to every vector in the row space of \( A \). Then in particular, \( \mathbf{v} \) is orthogonal to the row vectors \( \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m \) of \( A \); that is
  \[
  \mathbf{r}_1 \cdot \mathbf{v} = \mathbf{r}_2 \cdot \mathbf{v} = \cdots = \mathbf{r}_m \cdot \mathbf{v} = 0
  \]
Proof of Theorem 6.2.6(a)

\[ \mathbf{r}_1 \cdot \mathbf{v} = \mathbf{r}_2 \cdot \mathbf{v} = \cdots = \mathbf{r}_m \cdot \mathbf{v} = 0 \]

- The linear system \( \mathbf{A}\mathbf{x} = \mathbf{0} \) can be expressed in dot product notation as

\[
\begin{bmatrix}
\mathbf{r}_1 \cdot \mathbf{x} \\
\mathbf{r}_2 \cdot \mathbf{x} \\
\vdots \\
\mathbf{r}_m \cdot \mathbf{x}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

- And it follows that \( \mathbf{v} \) is a solution of this system and hence lies in the nullspace of \( \mathbf{A} \).
Proof of Theorem 6.2.6(a)

- Conversely, assume that \( \mathbf{v} \) is a vector in the nullspace of \( A \), so \( A\mathbf{v} = \mathbf{0} \). It follows that \( \mathbf{r}_1 \cdot \mathbf{v} = \mathbf{r}_2 \cdot \mathbf{v} = \cdots = \mathbf{r}_m \cdot \mathbf{v} = 0 \).

- But if \( \mathbf{r} \) is any vector in the row space of \( A \), then \( \mathbf{r} \) is expressible as a linear combination of the row vectors of \( A \), say \( \mathbf{r} = c_1 \mathbf{r}_1 + c_2 \mathbf{r}_2 + \cdots + c_m \mathbf{r}_m \).

- Thus \( \mathbf{r} \cdot \mathbf{v} = (c_1 \mathbf{r}_1 + c_2 \mathbf{r}_2 + \cdots + c_m \mathbf{r}_m) \cdot \mathbf{v} = c_1(\mathbf{r}_1 \cdot \mathbf{v}) + c_2(\mathbf{r}_2 \cdot \mathbf{v}) + \cdots + c_m(\mathbf{r}_m \cdot \mathbf{v}) = 0 + 0 + \cdots + 0 = 0 \).

- Which proves that \( \mathbf{v} \) is orthogonal to every vector in the row space of \( A \).
Example (Basis for an Orthogonal Complement)

- Let $W$ be the subspace of $\mathbb{R}^5$ spanned by the vectors $w_1=(2, 2, -1, 0, 1)$, $w_2=(-1, -1, 2, -3, 1)$, $w_3=(1, 1, -2, 0, -1)$, $w_4=(0, 0, 1, 1, 1)$. Find a basis for the orthogonal complement of $W$.

- Solution
  - The space $W$ spanned by $w_1$, $w_2$, $w_3$, and $w_4$ is the same as the row space of the matrix
    \[
    A = \begin{bmatrix}
    2 & 2 & -1 & 0 & 1 \\
    -1 & -1 & 2 & -3 & 1 \\
    1 & 1 & -2 & 0 & -1 \\
    0 & 0 & 1 & 1 & 1
    \end{bmatrix}
    \]
  - By Theorem 6.2.6, the nullspace of $A$ is the orthogonal complement of $W$.
  - In Example 4 of Section 5.5 we showed that
    \[
    \begin{bmatrix}
    -1 \\
    1 \\
    0 \\
    0 \\
    0
    \end{bmatrix}
    \quad \text{and} \quad
    \begin{bmatrix}
    -1 \\
    0 \\
    -1 \\
    0 \\
    1
    \end{bmatrix}
    \]
    form a basis for this nullspace.
  - Thus, vectors $v_1 = (-1, 1, 0, 0, 0)$ and $v_2 = (-1, 0, -1, 0, 1)$ form a basis for the orthogonal complement of $W$. 

Remarks

- In any inner product space $V$, the zero space $\{0\}$ and the entire space $V$ are orthogonal complements.
- If $A$ is an $n \times n$ matrix, to say that $Ax=0$ has only the trivial solution is equivalent to saying that the orthogonal complement of the nullspace of $A$ is all of $\mathbb{R}^n$, or equivalently, that the row space of $A$ is all of $\mathbb{R}^n$. 
Theorem 6.2.7 (Equivalent Statements)

If $A$ is an $m \times n$ matrix, and if $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is multiplication by $A$, then the following are equivalent:

- $A$ is invertible.
- $Ax = 0$ has only the trivial solution.
- The reduced row-echelon form of $A$ is $I_n$.
- $A$ is expressible as a product of elementary matrices.
- $Ax = b$ is consistent for every $n \times 1$ matrix $b$.
- $Ax = b$ has exactly one solution for every $n \times 1$ matrix $b$.
- $\det(A) \neq 0$.
- The range of $T_A$ is $\mathbb{R}^n$.
- $T_A$ is one-to-one.
- The column vectors of $A$ are linearly independent.
- The row vectors of $A$ are linearly independent.
- The column vectors of $A$ span $\mathbb{R}^n$.
- The row vectors of $A$ span $\mathbb{R}^n$.
- The column vectors of $A$ form a basis for $\mathbb{R}^n$.
- The row vectors of $A$ form a basis for $\mathbb{R}^n$.
- $A$ has rank $n$.
- $A$ has nullity 0.
- The orthogonal complement of the nullspace of $A$ is $\mathbb{R}^n$.
- The orthogonal complement of the row of $A$ is $\{0\}$. 