Lecture 12: 4.1~4.2 Euclidean Vector Space

Wei-Ta Chu

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Definitions

- If n is a positive integer, then an ordered n-tuple is a sequence of n real numbers $(a_1, a_2, \ldots, a_n)$. The set of all ordered n-tuple is called n-space and is denoted by R^n.
- (a_1, a_2, a_3) can be interpreted as a point or a vector in R^3.
- Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n are called equal if

$$u_1 = v_1, u_2 = v_2, \ \ldots, \ u_n = v_n$$

The sum $\mathbf{u} + \mathbf{v}$ is defined by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ u_1 + v_1, \ \ldots, \ u_n + v_n)$$

and if k is any scalar, the scalar multiple $k\mathbf{u}$ is defined by

$$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$$

Some Examples of Vectors

- Graphical images
  - One way to describe a color is by assigning each pixel three numbers that means the *red*, *blue*, and *green* of the pixel.
  - E.g.: red (255, 0, 0), yellow (255, 255, 0)
- Experimental data
  - A scientist performs an experiment and makes n numerical measurements each time. The result of each experiment can be regarded as a vector $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ in R^n in which $y_1,y_2,\ldots,y_n$ are the measured values.

Remarks

- The operations of addition and scalar multiplication in this definition are called the *standard operations* on R^n.

- The zero vector in R^n is denoted by $\mathbf{0}$ and is defined to be the vector $\mathbf{0} = (0, 0, \ldots, 0)$.
- If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is any vector in R^n, then the negative (or additive inverse) of $\mathbf{u}$ is denoted by $-\mathbf{u}$ and is defined by $-\mathbf{u} = (-u_1, -u_2, \ldots, -u_n)$.
- The difference of vectors in R^n is defined by

$$\mathbf{v} - \mathbf{u} = \mathbf{v} + (-\mathbf{u}) = (v_1 - u_1, v_2 - u_2, \ldots, v_n - u_n)$$

Theorem 4.1.1 (Properties of Vector in R^n)

- If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in R^n and k and l are scalars, then:
  - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
  - $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
  - $\mathbf{u} + (-\mathbf{u}) = 0$; that is $\mathbf{u} - \mathbf{u} = 0$
  - $k(l\mathbf{u}) = (kl)\mathbf{u}$
  - $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
  - $(k+l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
  - $1\mathbf{u} = \mathbf{u}$

Euclidean Inner Product

- Definition
  - If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the Euclidean inner product $\mathbf{u} \cdot \mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

- Example
  - The Euclidean inner product of the vectors $\mathbf{u} = (-1,3,5,7)$ and $\mathbf{v} = (5,-4,7,0)$ in R^4 is

$$\mathbf{u} \cdot \mathbf{v} = (-1)(5) + (3)(-4) + (5)(7) + (7)(0) = 18$$

Properties of Euclidean Inner Product

- Theorem 4.1.2
  - If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^n and k is any scalar, then
    - $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
    - $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
    - $(k\,\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$
    - $\mathbf{v} \cdot \mathbf{v} \geq \mathbf{0}$; Further, $\mathbf{v} \cdot \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{v} = 0$
- Example
  - $(3\mathbf{u} + 2\mathbf{v}) \cdot (4\mathbf{u} + \mathbf{v})$
    $= (3\mathbf{u}) \cdot (4\mathbf{u} + \mathbf{v}) + (2\mathbf{v}) \cdot (4\mathbf{u} + \mathbf{v})$
    $= (3\mathbf{u}) \cdot (4\mathbf{u}) + (3\mathbf{u}) \cdot \mathbf{v} + (2\mathbf{v}) \cdot (4\mathbf{u}) + (2\mathbf{v}) \cdot \mathbf{v}$
    $=12(\mathbf{u} \cdot \mathbf{u}) + 11(\mathbf{u} \cdot \mathbf{v}) + 2(\mathbf{v} \cdot \mathbf{v})$

Norm and Distance in Euclidean *n*-Space

- We define the Euclidean norm (or Euclidean length) of a vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n by

$$\|\mathbf{u}\| = (\mathbf{u} \cdot \mathbf{u})^{1/2} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

- Similarly, the Euclidean distance between the points $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\boldsymbol{R}^n$ is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$

- Example
  - If $\mathbf{u} = (1,3,-2,7)$ and $\mathbf{v} = (0,7,2,2)$, then in the Euclidean space R^4

$$\|\mathbf{u}\| = \sqrt{(1)^2 + (3)^2 + (-2)^2 + (7)^2} = \sqrt{63} = 3\sqrt{7}$$

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$

Theorems

- Theorem 4.1.3 (Cauchy-Schwarz Inequality in R^n)
  - If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

- For vectors in R^2 and R^3, we know

$$|\boldsymbol{u} \cdot \boldsymbol{v}| = |\|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos\theta| = \|\boldsymbol{u}\| \|\boldsymbol{v}\| |\cos\theta| \leq \|\boldsymbol{u}\| \|\boldsymbol{v}\|$$

Theorems

- Theorem 4.1.4 (Properties of Length in R^n)
  - If **u** and **v** are vectors in R^n and k is any scalar, then
    - $\| \mathbf{u} \| \geq 0$
    - $\| \mathbf{u} \| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
    - $\| k\mathbf{u} \| = | k | \| \mathbf{u} \|$
    - $\| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \|$ (Triangle inequality)

Proof of Theorem 4.1.4 (c)

$\| k\mathbf{u} \| = | k | \| \mathbf{u} \|$

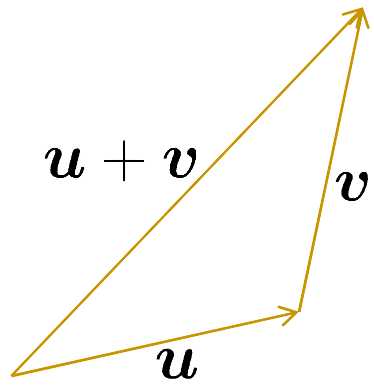
- If $\mathbf{u}=(u_1,u_2,\ldots,u_n)$, then $k\mathbf{u}=(ku_1,ku_2,\ldots,ku_n)$, so

$$\begin{aligned}
\|k\boldsymbol{u}\| &= \sqrt{(ku_1)^2 + (ku_2)^2 + \cdots + (ku_n)^2} \\
&= |k|\sqrt{(u_1)^2 + (u_2)^2 + \cdots + (u_n)^2} \\
&= |k|\|\boldsymbol{u}\|
\end{aligned}$$

Proof of Theorem 4.1.4 (d)

$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$

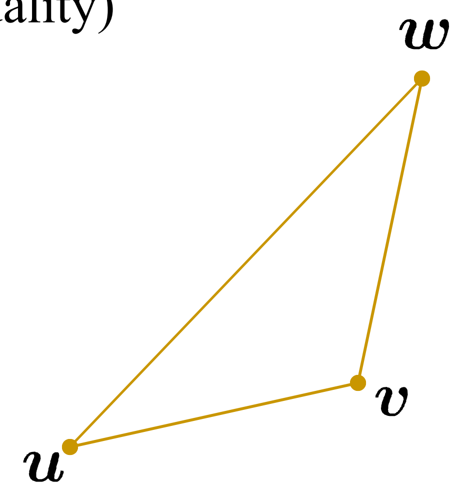
$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = (\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) = (\boldsymbol{u}\cdot\boldsymbol{u}) + 2(\boldsymbol{u}\cdot\boldsymbol{v}) + (\boldsymbol{v}\cdot\boldsymbol{v})$$
$$= \|\boldsymbol{u}\|^2 + 2(\boldsymbol{u}\cdot\boldsymbol{v}) + \|\boldsymbol{v}\|^2$$
$$\le \|\boldsymbol{u}\|^2 + 2|\boldsymbol{u}\cdot\boldsymbol{v}| + \|\boldsymbol{v}\|^2$$
$$\le \|\boldsymbol{u}\|^2 + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \|\boldsymbol{v}\|^2$$
$$= (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)^2$$



Theorems

- Theorem 4.1.5 (Properties of Distance in R^n)
  - If **u**, **v**, and **w** are vectors in R^n and k is any scalar, then
    - $d(\mathbf{u}, \mathbf{v}) \geq 0$
    - $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
    - $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
    - $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ (Triangle inequality)
- Proof (d)

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \|(\boldsymbol{u} - \boldsymbol{w}) + (\boldsymbol{w} - \boldsymbol{v})\|$$
$$\leq \|\boldsymbol{u} - \boldsymbol{w}\| + \|\boldsymbol{w} - \boldsymbol{v}\| = d(\boldsymbol{u}, \boldsymbol{w}) + d(\boldsymbol{w}, \boldsymbol{v})$$



Theorems

- Theorem 4.1.6
  - If **u**, **v**, and **w** are vectors in R^n with the Euclidean inner product, then

$$\mathbf{u} \cdot \mathbf{v} = \tfrac{1}{4} \| \mathbf{u} + \mathbf{v} \|^2 - \tfrac{1}{4} \| \mathbf{u} - \mathbf{v} \|^2$$

- Proof:

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 = (\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{u} + \boldsymbol{v}) = \|\boldsymbol{u}\|^2 + 2(\boldsymbol{u} \cdot \boldsymbol{v}) + \|\boldsymbol{v}\|^2$$

$$\|\boldsymbol{u} - \boldsymbol{v}\|^2 = (\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v}) = \|\boldsymbol{u}\|^2 - 2(\boldsymbol{u} \cdot \boldsymbol{v}) + \|\boldsymbol{v}\|^2$$

Orthogonality

- Definition
  - Two vectors **u** and **v** in R^n are called orthogonal if **u** · **v** = 0

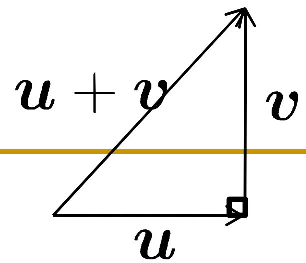
- Example
  - In the Euclidean space R^4 the vectors

    $$\mathbf{u} = (-2, 3, 1, 4) \text{ and } \mathbf{v} = (1, 2, 0, -1)$$

    are orthogonal, since **u** · **v** = (-2)(1) + (3)(2) + (1)(0) + (4)(-1) = 0

- Theorem 4.1.7 (Pythagorean Theorem in R^n)
  - If **u** and **v** are orthogonal vectors in R^n which the Euclidean inner product, then

    $$\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2$$



Matrix Formula for the Dot Product

- If we use column matrix notation for the vectors

$$\mathbf{u} = [u_1\ u_2\ \ldots\ u_n]^T \text{ and } \mathbf{v} = [v_1\ v_2\ \ldots\ v_n]^T,$$

or

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

then

$$\boldsymbol{v}^T \boldsymbol{u} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$$= \begin{bmatrix} u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{u} \cdot \boldsymbol{v} \end{bmatrix} = \boldsymbol{u} \cdot \boldsymbol{v}$$

Matrix Formula for the Dot Product

- For vectors in column matrix notation, we have the following formula for the Euclidean inner product:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T \mathbf{u}$$

- For example:

$$\boldsymbol{u} = \begin{bmatrix} -1 \\ 3 \\ 5 \\ 7 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 5 \\ -4 \\ 7 \\ 0 \end{bmatrix}$$

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v}^T \boldsymbol{u} = \begin{bmatrix} 5 & -4 & 7 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 18 \end{bmatrix} = 18$$

Matrix Formula for the Dot Product

- If A is an $n \times n$ matrix, then it follows

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T(A\mathbf{u}) = (\mathbf{v}^T A)\mathbf{u} = (A^T\mathbf{v})^T\mathbf{u} = \mathbf{u} \cdot A^T\mathbf{v}$$

$$\mathbf{u} \cdot A\mathbf{v} = (A\mathbf{v})^T\mathbf{u} = (\mathbf{v}^T A^T)\mathbf{u} = \mathbf{v}^T(A^T\mathbf{u}) = A^T\mathbf{u} \cdot \mathbf{v}$$

- These provide important links between multiplication by an $n \times n$ matrix A and multiplication by A^T.

Example

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

$$A\boldsymbol{u} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$$

$$A^T\boldsymbol{v} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix}$$

$$A\boldsymbol{u} \cdot \boldsymbol{v} = 7(-2) + 10(0) + 5(5) = 11$$

$$\boldsymbol{u} \cdot A^T\boldsymbol{v} = (-1)(-7) + 2(4) + 4(-1) = 11$$

A Dot Product View of Matrix Multiplication

- If $A = [a_{ij}]$ is an $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix, then the ij-th entry of AB is

  $$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{ir}b_{rj}$$

  which is the dot product of the ith row vector of A and the jth column vector of B
- Thus, if the row vectors of A are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ and the column vectors of B are $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$, then the matrix product AB can be expressed as

$$AB = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{c}_n \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{c}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_m \cdot \mathbf{c}_1 & \mathbf{r}_m \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_m \cdot \mathbf{c}_n \end{bmatrix}$$

A Dot Product View of Matrix Multiplication

- A linear system $A\mathbf{x}=\mathbf{b}$ can be expressed in dot product form as

$$\begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{x} \\ \boldsymbol{r}_2 \cdot \boldsymbol{x} \\ \vdots \\ \boldsymbol{r}_m \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Example

System

$$
\begin{aligned}
3x_1 - 4x_2 + x_3 &= 1 \\
2x_1 - 7x_2 - 4x_3 &= 5 \\
x_1 + 5x_2 - 8x_3 &= 0
\end{aligned}
$$

Dot Product Form

$$
\begin{bmatrix} (3,-4,1)\cdot(x_1,x_2,x_3) \\ (2,-7,-4)\cdot(x_1,x_2,x_3) \\ (1,5,-8)\cdot(x_1,x_2,x_3) \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}
$$

Functions from R^n to R

- A function is a rule f that associates with each element in a set A one and only one element in a set B.
- If f associates the element b with the element, then we write $b = f(a)$ and say that b is the image of a under f or that $f(a)$ is the value of f at a.
- The set A is called the domain of f and the set B is called the codomain of f.
- The subset of B consisting of all possible values for f as a varies over A is called the range of f.

Examples

Formula	Example	Classification	Description
$f(x)$	$f(x)=x^2$	Real-valued function of a real variable	Function from R to R
$f(x,y)$	$f(x,y)=x^2+y^2$	Real-valued function of two real variable	Function from R^2 to R
$f(x,y,z)$	$f(x,y,z)=x^2+y^2+z^2$	Real-valued function of three real variable	Function from R^3 to R
$f(x_1,x_2,\ldots,x_n)$	$f(x_1,x_2,\ldots,x_n)=x_1^2+x_2^2+\ldots+x_n^2$	Real-valued function of n real variable	Function from R^n to R

Function from R^n to R^m

- If the domain of a function f is R^n and the codomain is R^m, then f is called a map or transformation from R^n to R^m. We say that the function f maps R^n into R^m, and denoted by $f : R^n \to R^m$.

- If $m = n$ the transformation $f : R^n \to R^m$ is called an operator on R^n.

Function from R^n to R^m

- Suppose $f_1, f_2, \ldots, f_m$ are real-valued functions of n real variables, say

$$w_1 = f_1(x_1, x_2, \ldots, x_n)$$
$$w_2 = f_2(x_1, x_2, \ldots, x_n)$$
$$\cdots$$
$$w_m = f_m(x_1, x_2, \ldots, x_n)$$

  These m equations assign a unique point $(w_1, w_2, \ldots, w_m)$ in R^m to each point $(x_1, x_2, \ldots, x_n)$ in R^n and thus define a transformation from R^n to R^m. If we denote this transformation by $T: R^n \to R^m$ then

$$T(x_1, x_2, \ldots, x_n) = (w_1, w_2, \ldots, w_m)$$

Example: A Transformation from R^2 to R^3

$$w_1 = x_1 + x_2$$
$$w_2 = 3x_1x_2$$
$$w_3 = x_1^2 - x_2^2$$

- Define a transform T: $R^2 \longrightarrow R^3$
- With this transformation, the image of the point (x_1, x_2) is

$$T(x_1, x_2) = (x_1 + x_2, 3x_1x_2, x_1^2 - x_2^2)$$

- Thus, for example, T(1,-2) = (-1, -6, -3)

Linear Transformations from R^n to R^m

- A linear transformation (or a linear operator if $m = n$) $T: R^n \rightarrow R^m$ is defined by equations of the form

$$\begin{aligned} w_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ w_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ &\vdots \\ w_m &= a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{aligned} \quad \text{or} \quad \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{13} \\ a_{21} & a_{22} & \cdots & a_{23} \\ \vdots & \vdots & & \vdots \\ a_{mn} & a_{mn} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

or

$$\mathbf{w} = A\mathbf{x}$$

- The matrix $A = [a_{ij}]$ is called the standard matrix for the linear transformation T, and T is called multiplication by A.

Example (Transformation and Linear Transformation)

- The linear transformation $T : R^4 \to R^3$ defined by the equations

$$w_1 = 2x_1 - 3x_2 + x_3 - 5x_4$$
$$w_2 = 4x_1 + x_2 - 2x_3 + x_4$$
$$w_3 = 5x_1 - x_2 + 4x_3$$

the standard matrix for T (i.e., $\mathbf{w} = A\mathbf{x}$) is $A = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix}$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Notations

- Notations:
  - If it is important to emphasize that A is the standard matrix for T. We denote the linear transformation $T: R^n \rightarrow R^m$ by $T_A: R^n \rightarrow R^m$. Thus,

$$T_A(\mathrm{x}) = A\mathbf{x}$$

  - We can also denote the standard matrix for T by the symbol $[T]$, or

$$T(\mathbf{x}) = [T]\mathbf{x}$$

Remarks

- Remark:
  - We have establish a correspondence between $m \times n$ matrices and linear transformations from R^n to R^m :
    - To each matrix A there corresponds a linear transformation T_A (multiplication by A), and to each linear transformation $T: R^n \to R^m$, there corresponds an $m \times n$ matrix $[T]$ (the standard matrix for T).

Zero Transformation

- Zero Transformation from R^n to R^m
  - If 0 is the $m \times n$ zero matrix and $\mathbf{0}$ is the zero vector in R^n, then for every vector $\mathbf{x}$ in R^n

$$T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$$

  - So multiplication by zero maps every vector in R^n into the zero vector in R^m. We call T_0 the zero transformation from R^n to R^m.

Identity Operator

- Identity Operator on R^n
  - If I is the $n \times n$ identity, then for every vector in R^n

$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

  - So multiplication by I maps every vector in R^n into itself.
  - We call T_I the identity operator on R^n.

Reflection Operators

- In general, operators on R^2 and R^3 that map each vector into its symmetric image about some line or plane are called reflection operators.
- Such operators are linear.

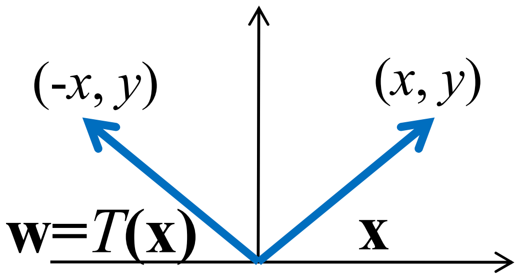
Example

- If we let **w**=T(**x**), then the equations relating the components of **x** and **w** are

$$w_1 = -x = -x + 0y$$
$$w_2 = y = 0x + y$$

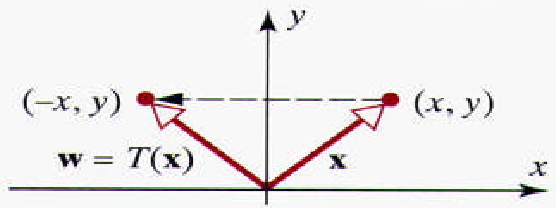
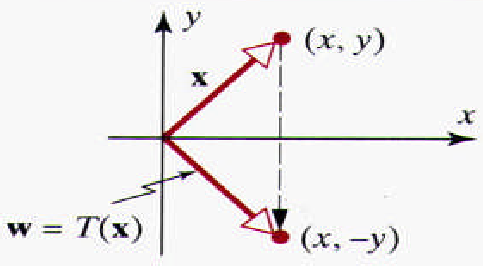
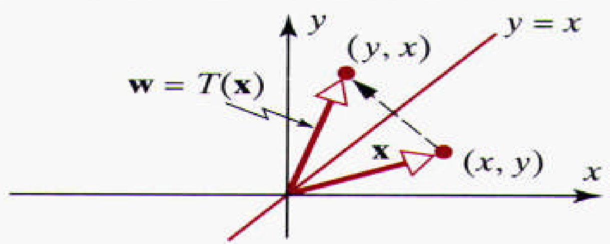
or, in matrix form

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

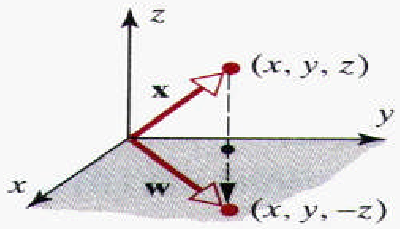
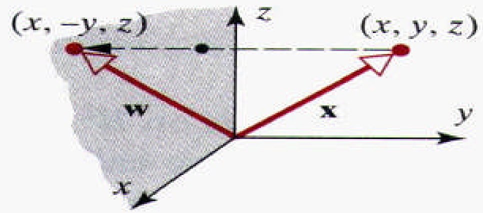
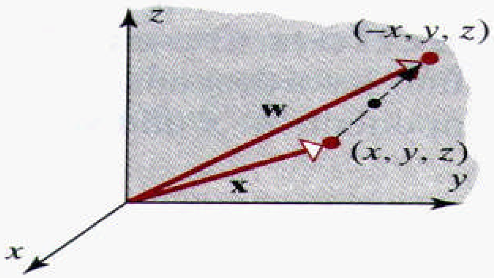


- The standard matrix for T is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Reflection Operators (2-Space)

Operator	Illustration	Equations	Standard Matrix
Reflection about the y-axis		$w_1 = -x$ $w_2 = y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the x-axis		$w_1 = x$ $w_2 = -y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the line $y = x$		$w_1 = y$ $w_2 = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Reflection Operators (3-Space)

Operator	Illustration	Equations	Standard Matrix
Reflection about the xy-plane		$w_1 = x$ $w_2 = y$ $w_3 = -z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane		$w_1 = x$ $w_2 = -y$ $w_3 = z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane		$w_1 = -x$ $w_2 = y$ $w_3 = z$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$