Image Registration Lecture 3: Images and Transformations

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Outline

- Images:
  - Types of images
  - Coordinate systems
- Transformations
  - Similarity transformations in 2d and 3d
  - Affine transformations in 2d and 3d
  - Projective transformations
  - Non-linear and deformable transformations

Images

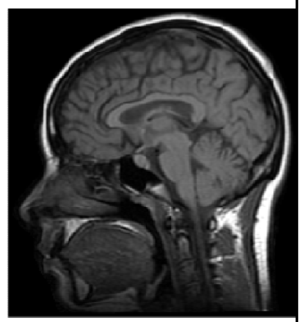
- Medical (tomographic) images:
  - CT and MRI
- Intensity / video images
- Range images

CT: X-Ray Computed Tomography

- 電腦斷層
- X-ray projection through body
- Reconstruction of interior via computed tomography
- A volume of slices
  - Voxels in each slice are between 0.5 mm and 1.0 mm on a side
  - Spacing between slices is typically 1.0 and 5.0 mm
- Resulting intensities are measured in Hounsfield units
  - 0 for water
  - ~ -1000 for air, +1000 for bone

MRI: Magnetic Resonance Images

- 磁核共振影像
- Magnetic fields:
  - Nuclei of individual water molecules
  - Induced fields
- Measure the relaxation times of molecules as magnetic fields are changed
- Images are recorded as individual slices and as volumes
  - Slices need not be axial as in CT
- MRIs are subject to field distortions
- There are many different types of MRIs



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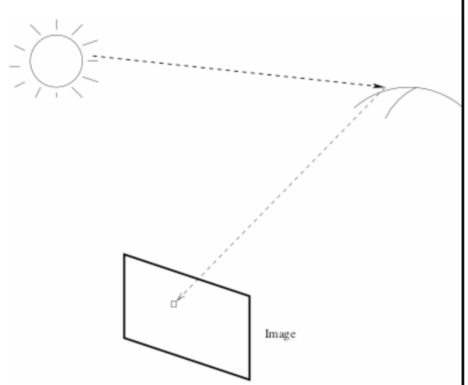
Video Images

- Pixel intensities depend on
  - Illumination
  - Surface orientation
  - Reflectance
  - Viewing direction projection
  - Digitization
- By themselves
  - Pixel dimensions have no physical meaning
  - Pixel intensities have no physical meaning



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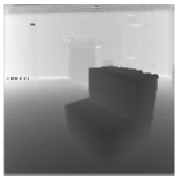
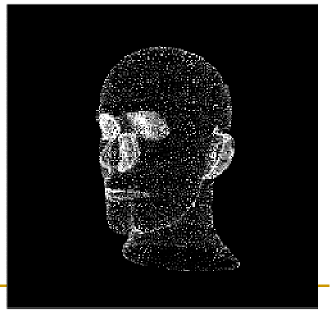
Range Images

- Measuring depth
- Representation:
  - (x,y,z) at each pixel
  - "Point cloud" of (x,y,z) values



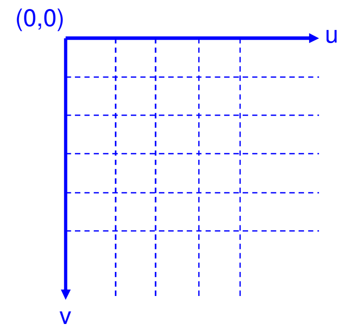


Coordinate Systems

- Topics:
  - Image coordinates
  - Physical coordinates
  - Mapping between them
- One must be aware of these, especially in the implementation details of a registration algorithm

Image Coordinate Systems

- The origin is generally at the corner of the image, often the upper left
- Directions:
  - *x* axis is across
  - *y* axis is down
- Differs from matrix coordinates and Cartesian coordinates
- Units are integer increments of grid indices



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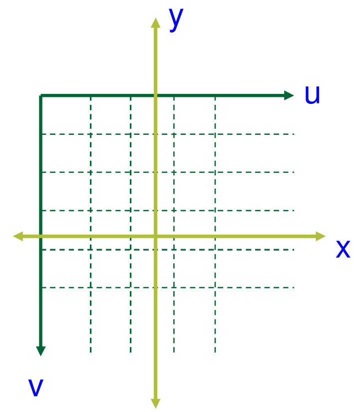
Physical Coordinates

- Origin:
  - Near center in video images
  - Defined by scanner for medical images
- Directions:
  - Right-handed coordinate frame
- Units are millimeters (medical images)



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Non-Isotropic Voxels in Medical Images

- Axial (*z*) dimension is often greater than within slice (*x* and *y*) dimensions
- At worst this can be close to an order of magnitude
- New scanners are getting close to isotropic dimensions

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Change of Coordinates as Matrix Multiplication

- Physical coordinates to pixel coordinates:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & -s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_u \\ t_v \end{pmatrix}$$

Scaling from physical to pixel units: parameters are pixels per mm

Translation of origin in pixel units

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Homogeneous Form

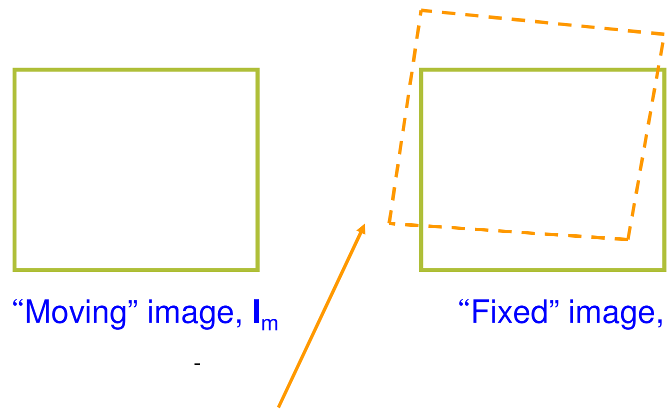
- Using homogeneous coordinates --- in simplest terms, adding a 1 at the end of each vector --- allows us to write this using a single matrix multiplication:

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & t_u \\ 0 & -s_y & t_v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Transformations

- Geometric transformations
- Intensity transformations

Forward Geometric Transformation



"Moving" image, $\mathbf{I}_m$

"Fixed" image, $\mathbf{I}_f$

Transformed moving image, $\mathbf{I}_m'$. It is mapped into the coordinate system of $\mathbf{I}_f$

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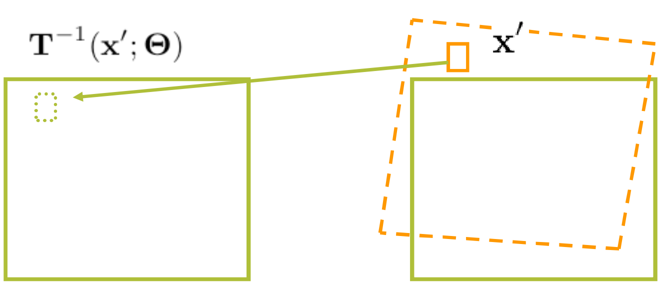
Mapping Pixel Values - Going Backwards



"Moving" image, $\mathbf{I}_m$

"Fixed" image, $\mathbf{I}_f$

$$I'_m(\mathbf{x}') = I_m(\mathbf{T}^{-1}(\mathbf{x}';\mathbf{\Theta}))$$

Need to go backward to get pixel value. This will generally not "land" on a single pixel location. Instead you need to do interpolation.

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Backward Transformation

- Because of the above, some techniques estimate the parameters of the "backwards" transformation, $\mathbf{T}^{-1}$:
  - From the fixed image to the moving image
- This can sometimes become a problem if the mapping function is non-invertible
- Intensity-based algorithms generally estimate the backwards transformation

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Intensity Transformations As Well

- When the intensities must be transformed as well, the equations get more complicated:

$$I'_m(\mathbf{x}') = \mathbf{S}(I_m(\mathbf{T}^{-1}(\mathbf{x}'; \boldsymbol{\Theta})); \gamma)$$

Intensity mapping function ($\mathbf{S}$)

Intensity mapping parameters (γ)

Fortunately, we will not be very concerned with intensity mapping

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Transformation Models - A Start

- Translation and scaling in 2d and 3d
- Similarity transformations in 2d and 3d
- Affine transformations in 2d and 3d

Translation and Independent Scaling

- We've already seen this in 2d:
  - Transformations between pixel and physical coordinates
  - Translation and scaling in the retina application
- In matrix form: $\mathbf{T}(\mathbf{x};\mathbf{\Theta}) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \mathbf{x} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$
- Where $\mathbf{\Theta} = \begin{pmatrix} s_x & s_y & t_x & t_y \end{pmatrix}^T$
- In homogeneous form: $\mathbf{T}(\mathbf{x};\mathbf{\Theta}) = \begin{pmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}$

Two-component vector

Similarity Transformations

- Components:
  - Rotation (next slide)
  - Translation
  - independent scaling
- "Invariants"
  - Angles between vectors remain fixed
  - All lengths scale proportionally, so the ratio of lengths is preserved
- Example applications:
  - Multimodality, intra-subject (same person) brain registration
  - Registration of range images (no scaling)

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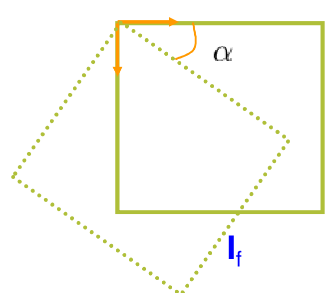
Rotations in 2d

- Assume (for now) that forward transformation is only a rotation
- Image origins are in upper left corner; angles are measured "clockwise" with respect to *x* axis



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

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Similarity Transformations in 2d

- As a result:

$$\mathbf{T}(\mathbf{x};\boldsymbol{\Theta}) = s\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\mathbf{x} + \begin{pmatrix}t_x \\ t_y\end{pmatrix}$$

$$\boldsymbol{\Theta} = \begin{pmatrix}s & \alpha & t_x & t_y\end{pmatrix}^T$$

- Or:

$$\mathbf{T}(\mathbf{x};\boldsymbol{\Theta}) = \begin{pmatrix}s\cos\alpha & -s\sin\alpha \\ s\sin\alpha & s\cos\alpha\end{pmatrix}\mathbf{x} + \begin{pmatrix}t_x \\ t_y\end{pmatrix} = \begin{pmatrix}a & -b \\ b & a\end{pmatrix}\mathbf{x} + \begin{pmatrix}t_x \\ t_y\end{pmatrix}$$

$$\boldsymbol{\Theta} = \begin{pmatrix}a & b & t_x & t_y\end{pmatrix}^T$$

Rotation Matrices and 3d Rotations

- The 2d rotation matrix is orthonormal with determinant 1:

$$\mathbf{R} = \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}$$

- In arbitrary dimensions, these two properties hold.
- Therefore, we write similarity transformations in arbitrary dimensions as

$$\mathbf{T}(\mathbf{x};\boldsymbol{\Theta}) = s\mathbf{R}\mathbf{x} + \mathbf{t} \qquad \mathbf{T}(\mathbf{x};\boldsymbol{\Theta}) = \begin{pmatrix}s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1\end{pmatrix}\begin{pmatrix}\mathbf{x} \\ 1\end{pmatrix}$$

Representing Rotations in 3d (Teaser)

- **R** is a 3x3 matrix, giving it 9 parameters, but it has only 3 degrees of freedom
  - Its orthonormality properties remove the other 6.
- Question: How to represent **R**?
- Some answers (out of many):
  - Small angle approximations about each axis
  - Quaternions

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$$\mathbf{R} = \mathbf{R}_x\mathbf{R}_y\mathbf{R}_z$$
$$= \begin{pmatrix} \cos\alpha\cos\theta & -\cos\alpha\sin\theta & \sin\alpha \\ \cos\theta\sin\alpha\sin\phi + \cos\phi\sin\theta & -\sin\alpha\sin\phi\sin\theta + \cos\phi\cos\theta & -\cos\alpha\sin\phi \\ -\cos\phi\cos\theta\sin\alpha + \sin\phi\sin\theta & \cos\phi\sin\alpha\sin\theta + \cos\theta\sin\phi & \cos\alpha\cos\phi \end{pmatrix}$$

Small angle approximation:

$$\sin\theta = \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \ldots \qquad \cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4} - \ldots$$
$$\approx \theta \qquad\qquad \approx 1$$

$$\mathbf{R} \approx \begin{pmatrix} 1 & -\theta & \alpha \\ \theta & 1 & -\phi \\ -\alpha & \phi & 1 \end{pmatrix} = \mathbf{I} - \begin{pmatrix} 0 & -\theta & \alpha \\ \theta & 0 & -\phi \\ -\alpha & \phi & 0 \end{pmatrix}$$

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Affine Transformations

- Geometry:
  - Rotation, scaling (each dimension separately), translation and shearing
- Invariants:
  - Parallel lines remain parallel
  - Ratio of lengths of segments along a line remains fixed.
- Example application:
  - Mosaic construction for images of earth surface (flat surface, relatively non-oblique cameras)

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Affine Transformations in 2d

- In 2d, multiply **x** location by arbitrary, but invertible 2x2 matrix:

$$\mathbf{T}(\mathbf{x};\mathbf{\Theta}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{x} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

- In general, $\mathbf{T}(\mathbf{x};\mathbf{\Theta}) = \mathbf{A}\mathbf{x} + \mathbf{t}$
- Or, in homogeneous form,

$$\mathbf{T}(\mathbf{x};\mathbf{\Theta}) = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}$$

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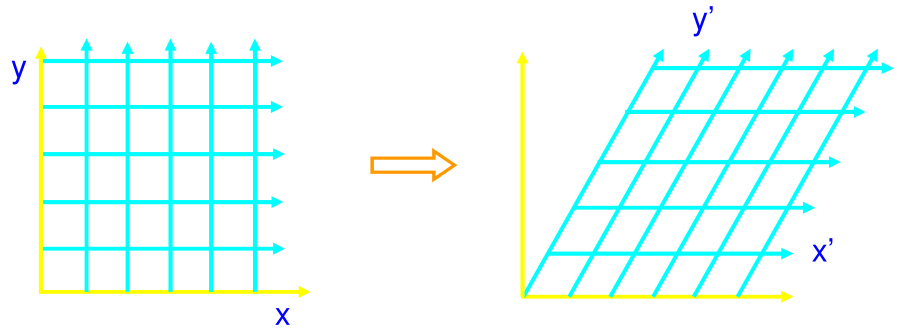
Affine Transformations in 2d



$$T(x;\Theta) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} x \quad \text{OR} \quad T(x;\Theta) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} x$$

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Projective Transformations

- General linear transformation
- Lengths, angle, and parallel lines are NOT preserved
- Coincidence, tangency, and complicated ratios of lengths are the invariants
- Application:
  - Mosaics of image taken by camera rotating about its optical center
  - Mosaic images of planar surface

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Algebra of Projective Transformations

- Add non-zero bottom row to the homogeneous matrix:

$$\mathbf{T}(\mathbf{x};\mathbf{\Theta}) = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^T & a_{nn} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}$$

- Transformed point location:

$$\mathbf{x}' = \frac{\mathbf{A}\mathbf{x} + \mathbf{t}}{\mathbf{v}^T\mathbf{x} + a_{nn}}$$

- Estimation is somewhat more complicated than for affine transformations

Non-Linear Transformations

- Transformation using in retinal image mosaics is a non-linear function of the image coordinates:

$$\mathbf{T}(\mathbf{x};\mathbf{\Theta}) = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta 13 & \theta_{14} & \theta_{15} & \theta 16 \\ \theta_{21} & \theta_{22} & \theta 23 & \theta_{24} & \theta_{25} & \theta 26 \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \\ x \\ y \\ 1 \end{pmatrix}$$

Deformable Models

- Non-global functions
- Often PDE-based, e.g.:
  - Thin-plate splines
  - Fluid mechanics
- Representation
  - B-splines
  - Finite elements
  - Fourier bases
  - Radial basis functions
- Lectures 27-28 will provide an introduction

Summary

- Images and image coordinate systems
  - Watch out for the difference between physical and pixel/voxel coordinates and for non-isotropic voxels!
- Transformation models
  - Forward and backward mapping
  - Our focus will be almost entirely geometric transformation as opposed to intensity transformations
- Transformations:
  - Similarity
  - Affine
  - Projective
  - Non-linear and deformable

Homework

- Available online