Image Registration Lecture 2: Vectors and Matrices

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Lecture Overview

- Vectors
- Matrices
  - Basics
  - Orthogonal matrices
  - Singular Value Decomposition (SVD)

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Preliminary Comments

- Some of this should be review; all of it might be review
- This is really only background, and not a main focus of the course
- All of the material is covered in standard linear algebra texts.
  - *Linear Algebra and Its Applications,* by Gilbert Strang
  - *Matrix Analysis and Applied Linear Algebra*, by C. D. Meyer, Carl Meyer

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Vectors: Definition

- Formally, a vector is an element of a vector space
- Informally (and somewhat incorrectly), we will use vectors to represent both point locations and directions
- Algebraically, we write
- Note that we will usually treat vectors column vectors and use the transpose notation to make the writing more compact

$$x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

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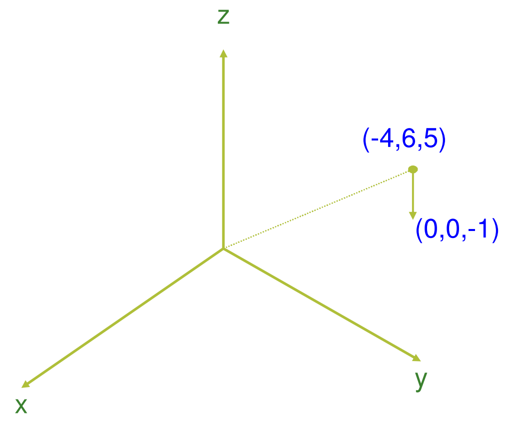
Vectors: Example



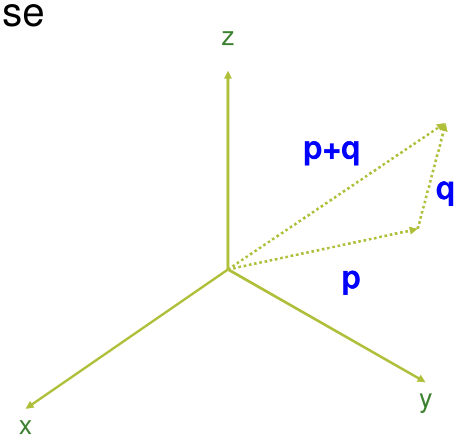
Vectors: Addition

- Added component-wise

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

- Example:

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$$



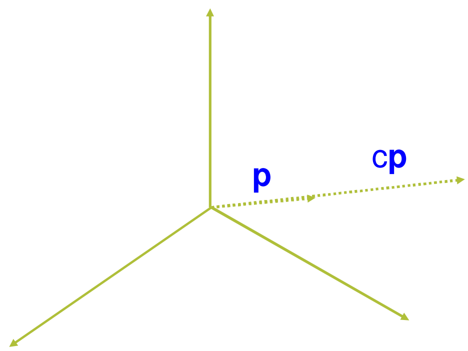
Vectors: Scalar Multiplication

- Simplest form of multiplication involving vectors
- In particular:

$$c\mathbf{x} = c\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix}$$

- Example:

$$3\begin{pmatrix} 0.5 \\ -3 \\ 8 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -9 \\ 24 \end{pmatrix}$$



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Vectors: Lengths, Magnitudes, Distances

- The length or magnitude of a vector is

$$\|\mathbf{x}\| = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2} = \sqrt{x_1^2 + \ldots + x_n^2}$$

- The distance between two vectors is

$$\|\mathbf{x} - \mathbf{y}\| = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$

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Vectors: Dot (Scalar/Inner) Product

- Second means of multiplication involving vectors
- In particular,

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i = (x_1 y_1 + \ldots + x_n y_n)$$

- We'll see a different notation for writing the scalar product using matrix multiplication soon
- Note that

$$\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$$

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Unit Vectors

- A unit (direction) vector is a vector whose magnitude is 1:

$$\|\mathbf{x}\| = \mathbf{x} \cdot \mathbf{x} = 1$$

- Typically, we will use a "hat" to denote a unit vector, e.g.: $\hat{\eta}$ $\hat{\mathbf{d}}$

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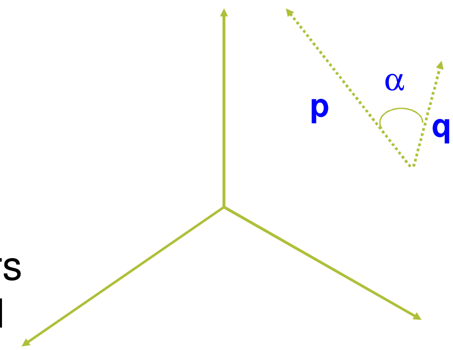
Angle Between Vectors

- We can compute the angle between two vectors using the scalar product:

$$\cos\alpha = \frac{\mathbf{p}\cdot\mathbf{q}}{\|\mathbf{p}\|\ \|\mathbf{q}\|}$$

- Two non-zero vectors are orthogonal if and only if $\mathbf{p}\cdot\mathbf{q} = 0$



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Cross (Outer) Product of Vectors

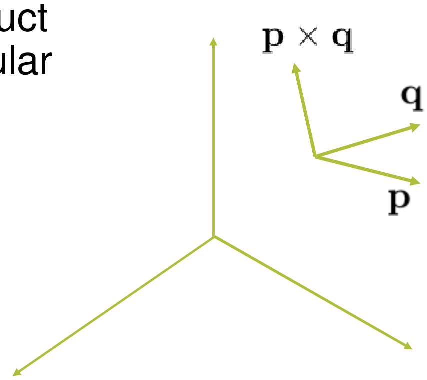
- Given two 3-vectors, **p** and **q**, the cross product is a vector perpendicular to both

$$\mathbf{r} = \mathbf{p}\times\mathbf{q}$$

- In component form,

$$\mathbf{r} = \begin{pmatrix} p_2q_3 - p_3q_2 \\ p_3q_1 - p_1q_3 \\ p_1q_2 - p_2q_3 \end{pmatrix}$$

- Finally, $\|\mathbf{p}\times\mathbf{q}\| = \sin\alpha\ \|\mathbf{p}\|\ \|\mathbf{q}\|$



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Looking Ahead A Bit to Transformations

- Be aware that lengths and angles are preserved by only very special transformations
- Therefore, in general
  - Unit vectors will no longer be unit vectors after applying a transformation
  - Orthogonal vectors will no longer be orthogonal after applying a transformation

Matrices - Definition

- Matrices are rectangular arrays of numbers, with each number subscripted by two indices:

$$\mathbf{A} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{pmatrix} a \\ a \\ \cdot \\ \cdot \\ \cdot \\ a \end{pmatrix}}\right\} m \text{ rows}$$

- A short-hand notation for this is $\mathbf{A} = (a_{ij})$

Special Matrices: The Identity

- The *identity matrix,* denoted I, I_n or $\mathbf{I}_{n \times n}$, is a square matrix with *n* rows and columns having 1's on the *main diagonal* and 0's everywhere else:

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{I}_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Diagonal Matrices

- A *diagonal matrix* is a square matrix that has 0's everywhere except on the main diagonal.
- For example:

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \underbrace{\mathrm{diag}}(2, -1, 3, 0)$$

Notational short-hand

Matrix Transpose and Symmetry

- The transpose of a matrix is one where the rows and columns are reversed:

$$\mathbf{A}^T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

- If **A** = **A**^{T} then the matrix is *symmetric*.
  - Only square matrices (*m*=*n*) are symmetric

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Examples

- This matrix is not symmetric

$$\begin{pmatrix} 3 & 4 & 5 & -1 \\ 4 & -1 & 6 & -4 \\ 5 & -1 & 0 & 0 \end{pmatrix}$$

- This matrix is symmetric

$$\begin{pmatrix} 3 & 4 & 5 & -1 \\ 4 & -1 & 6 & -4 \\ 5 & 6 & 0 & 0 \\ -1 & -4 & 0 & 1 \end{pmatrix}$$

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Matrix Addition

- Two matrices can be added if and only if (iff) they have the same number of rows and the same number of columns.
- Matrices are added component-wise:

$$\mathbf{A} + \mathbf{B} = (a_{i,j} + b_{i,j})$$

- Example:

$$\begin{pmatrix} 3 & 4 & 5 \\ 4 & -1 & 6 \end{pmatrix} + \begin{pmatrix} 5 & -2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 2 & 5 \\ 3 & -1 & 7 \end{pmatrix}$$

Matrix Scalar Multiplication

- Any matrix can be multiplied by a scalar

$$c\mathbf{A} = \mathbf{A}c = (ca_{ij})$$

Matrix Multiplication

- The *product* of an *m*x*n* matrix and a *n*x*p* matrix is a *m*x*p* matrix:

$$\mathbf{AB} = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)$$

- Entry *i,j* of the result matrix is the dot-product of row *i* of **A** and column *j* of **B**
- Example $\begin{pmatrix} 2 & 2 & 5 \\ 4 & -1 & 0 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ 21 & -8 \end{pmatrix}$

Vectors as Matrices

- Vectors, which we usually write as column vectors, can be thought of as *n*x*1* matrices
- The transpose of a vector is a *1*x*n* matrix - a row vector.
- These allow us to write the scalar product as a matrix multiplication: $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{a}$
- For example, $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = -5$

Notation

- We will tend to write matrices using boldface capital letters $\mathbf{A}$ $\mathbf{B}$
- We will tend to write vectors as boldface small letters $\mathbf{p}$ $\mathbf{x}$ $\mathbf{z}$

Square Matrices

- Much of the remaining discussion will focus only on square matrices:
  - Trace
  - Determinant
  - Inverse
  - Eigenvalues
  - Orthogonal / orthonormal matrices
- When we discuss the *singular value decomposition* we will be back to non-square matrices

Trace of a Matrix

- Sum of the terms on the main diagonal of a square matrix:

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{i,i}$$

- The trace equals the sum of the eigenvalues of the matrix.

Determinant

- Notation: $\det(\mathbf{A}) = |\mathbf{A}|$
- Recursive definition:
  - When *n=1*, $\det(\mathbf{A}) = a_{1,1}$
  - When *n=2*, $\det(\mathbf{A}) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$

Determinant (continued)

- For $n>2$, choose any row i of **A,** and define $\mathbf{M}_{i,j}$ be the $(n-1)$x$(n-1)$ matrix formed by deleting row i and column j of **A**, then

$$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\mathbf{M}_{ij})$$

- We get the same formula by choosing any column j of **A** and summing over the rows.

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Some Properties of the Determinant

- If any two rows or any two columns are equal, the determinant is 0
- Interchanging two rows or interchanging two columns reverses the sign of the determinant
- The determinant of **A** equals the product of the eigenvalues of **A**
- For square matrices $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

$$\det(\mathbf{A}) = \det(\mathbf{A}^T)$$

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Matrix Inverse

- The inverse of a square matrix **A** is the unique matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$$

- Matrices that do not have an inverse are said to be non-invertible or *singular*
- A matrix is invertible if and only if its determinant is non-zero
- We will not worry about the mechanism of calculating inverses, except using the singular value decomposition

Eigenvalues and Eigenvectors

- A scalar λ and a vector **v** are, respectively, an eigenvalue and an associated (unit) eigenvector of square matrix **A** if $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
- For example, if we think of a **A** as a transformation and if $\lambda=1$, then **Av**=**v** implies **v** is a "fixed-point" of the transformation.
- Eigenvalues are found by solving the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

- Once eigenvalues are known, eigenvectors are found, by finding the nullspace (we will not discuss this) of $\mathbf{A} - \lambda\mathbf{I}$

Eigenvalues of Symmetric Matrices

- They are all real (as opposed to imaginary), which can be seen by studying the following (and remembering properties of vector magnitudes)

$$\|\mathbf{A}\mathbf{v}\|^2 = \mathbf{v}^T\mathbf{A}^T\mathbf{A}\mathbf{v} = \mathbf{v}^T\mathbf{A}^T(\lambda\mathbf{v}) = \lambda\mathbf{v}^T\mathbf{A}\mathbf{v} = \lambda^2\mathbf{v}^T\mathbf{v}$$

- We can also show that eigenvectors associated with *distinct* eigenvalues of a symmetric matrix are *orthogonal*
- We can therefore write a symmetric matrix as

$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^T$$

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Orthonormal Matrices

- A square matrix is orthonormal (sometimes called orthogonal) iff $\mathbf{A}\mathbf{A}^T = \mathbf{I}$
  - In other word $\mathbf{A}^T$ is the right inverse.
- Based on properties of inverses this immediately implies $\mathbf{A}^T\mathbf{A} = \mathbf{I}$
- This means for vectors formed by any two rows or any two columns

$$\mathbf{a}_i^T\mathbf{a}_j = \delta_{i,j}$$

Kronecker delta, which is 1 if *i*=*j* and 0 otherwise

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Orthonormal Matrices - Properties

- The determinant of an orthonormal matrix is either 1 or -1 because

$$\det(\mathbf{Q}\mathbf{Q}^T) = \det(\mathbf{Q})\det(\mathbf{Q}^T) = \det(\mathbf{I}) = 1$$

- Multiplying a vector by an orthonormal matrix does not change the vector's length:

$$\|\mathbf{Q}\mathbf{v}\|^2 = \mathbf{v}^T\mathbf{Q}^T\mathbf{Q}\mathbf{v} = \mathbf{v}^T\mathbf{I}\mathbf{v} = \mathbf{v}^T\mathbf{v} = \|\mathbf{v}\|^2$$

- An orthonormal matrix whose determinant is 1 (-1) is called a rotation (reflection).
- Of course, as discussed on the previous slide

$$\mathbf{Q}^{-1} = \mathbf{Q}^T$$

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Singular Value Decomposition (SVD)

- Consider an *m*x*n* matrix, **A**, and assume *m*≥*n*.
- **A** can be "decomposed" into the product of 3 matrices:

$$\mathbf{A} = \mathbf{U}\mathbf{W}\mathbf{V}^T$$

- Where:
  - **U** is *m*x*n* with orthonormal columns
  - **W** is a *n*x*n* diagonal matrix of "singular values", and
  - **V** is *n*x*n* orthonormal matrix
- If *m*=*n* then **U** is an orthonormal matrix

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Properties of the Singular Values

$$\mathbf{W} = \mathrm{diag}(w_1, w_2, \ldots, w_n)$$

- with

$$w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$$

- and
  - the number of non-zero singular values is equal to the rank of **A**

SVD and Matrix Inversion

- For a non-singular, square matrix, with

$$\mathbf{A} = \mathbf{U}\mathbf{W}\mathbf{V}^T$$

- The inverse of **A** is

$$\mathbf{A} = \mathbf{V}\,\mathrm{diag}(1/w_1, 1/w_2, \ldots, 1/w_n)\mathbf{U}^T$$

  - You should confirm this for yourself!
- Note, however, this isn't always the best way to compute the inverse

SVD and Solving Linear Systems

- Many times problems reduce to finding the vector **x** that minimizes

$$\|\mathbf{Ax} - \mathbf{b}\|^2$$

- Taking the derivative (I don't necessarily expect that you can do this, but it isn't hard) with respect to **x**, setting the result to **0** and solving implies $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$
- Computing the SVD of **A** (assuming it is full-rank) results in $\mathbf{x} = \mathbf{VW}^{-1}\mathbf{U}^T\mathbf{b}$

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Summary

- Vectors
  - Definition, addition, dot (scalar / inner) product, length, etc.
- Matrices
  - Definition, addition, multiplication
  - Square matrices: trace, determinant, inverse, eigenvalues
  - Orthonormal matrices
  - SVD

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Looking Ahead to Lecture 3

- Images and image coordinate systems
- Transformations
  - Similarity
  - Affine
  - Projective