Image Registration Techniques
Homework 2
Due: Wednesday March 1st at the start of class

This homework explores Lectures 2 and 3. Answer each of the following questions clearly. You may submit hand-written answers, but if your handwriting isn't clear, please typeset your solutions.

1. (**5 points**) Given two vectors $\mathbf{v}$ and $\mathbf{u}$. Find the two components of $\mathbf{v}$, one in the direction of $\mathbf{u}$ and one perpendicular to $\mathbf{u}$.

2. (**10 points**) Show that if $\|\mathbf{A}\mathbf{v}\| = \|\mathbf{v}\|$ for all vectors $\mathbf{v}$ then $\mathbf{A}$ must be orthonormal.

   Solution: Writing the square magnitude using matrix multiplication,

   $$\mathbf{v}^T\mathbf{A}^T\mathbf{A}\mathbf{v} = \mathbf{v}^T\mathbf{v}.$$

   Re-arranging and factoring yields

   $$\mathbf{v}^T(\mathbf{A}^T\mathbf{A} - \mathbf{I})\mathbf{v} = 0.$$

   The only way this can be true of all vectors $\mathbf{v}$ is if

   $$\mathbf{A}^T\mathbf{A} - \mathbf{I} = \mathbf{0},$$

   or

   $$\mathbf{A}^T\mathbf{A} = \mathbf{I}.$$

3. (**10 points**) Find the inverse of the mapping from physical to pixel coordinates. Write it using matrix and vector notation in both homogeneous and non-homogeneous forms.

   Solution: Homogeneous form:

   $$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1/s_x & 0 & t_u/s_x \\ 0 & -1/s_y & -t_v/s_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

   Inhomogeneous forms:

   $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/s_x & 0 \\ 0 & -1/s_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} t_u/s_x \\ -t_v/s_y \end{pmatrix}$$

4. (**5 points**) Re-examine the algebraic formulation of the 2d similarity transformation from Lecture 3. Write parameters s and α as a function of a and b and write parameters a and b in terms of s and α. This will show that you can easily move back-and-forth between representations.

   Solution:

$$a = s \cos \alpha$$
$$b = s \sin \alpha$$

   and

$$s = \sqrt{a^2 + b^2}$$
$$\alpha = \tan^{-1} b/a$$

5. (**5 points**) Find the inverse of the similarity transformation,

$$\mathbf{x}' = s\mathbf{R}\mathbf{x} + \mathbf{t}$$

   and show that it is a similarity transformation. (You need only show that it has the form of a similarity transformation.)

   Solution:

$$\mathbf{x} = \frac{1}{s}\mathbf{R}^T(\mathbf{x}' - \mathbf{t}) = s'\mathbf{R}'\mathbf{x}' + \mathbf{t}'$$

   where

$$s' = \frac{1}{s}$$
$$\mathbf{R}' = \mathbf{R}^T$$
$$\mathbf{t}' = -\frac{1}{s}\mathbf{R}^T\mathbf{t}$$

   This has the properties of a similarity transformation because $\mathbf{R}^T$ is orthonormal and $s \neq 0$.

6. (**10 points**) Prove that the eigenvectors associated with distinct eigenvalues of a symmetric matrix are orthogonal. **Hint:** let $\mathbf{A}$ be a symmetric matrix, let λ_1 and λ_2 be eigenvalues with $\lambda_1 \neq \lambda_2$, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the associated eigenvectors; then consider the expression $\mathbf{v}_1^T\mathbf{A}\mathbf{v}_2$.

   Solution: Evaluate the expression in two ways:

$$\mathbf{v}_1^T\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1^T\lambda_2\mathbf{v}_2 = \lambda_2\mathbf{v}_1^T\mathbf{v}_2$$

   and

$$\mathbf{v}_1^T\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1^T\mathbf{A}^T\mathbf{v}_2 = (\mathbf{A}\mathbf{v}_1)^T\mathbf{v}_2 = \lambda_1\mathbf{v}_1^T\mathbf{v}_2$$

   Equating these, we have

$$\lambda_2\mathbf{v}_1^T\mathbf{v}_2 = \lambda_1\mathbf{v}_1^T\mathbf{v}_2.$$

   Since $\lambda_1 \neq \lambda_2$, the only way these can be equal is if $\mathbf{v}_1^T\mathbf{v}_2 = 0$. This means, of course, that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.