**Q2:** Writing the square magnitude using matrix multiplication,

\[ \mathbf{v}^T \mathbf{A}^T \mathbf{A} \mathbf{v} = \mathbf{v}^T \mathbf{v}. \]

Re-arranging and factoring yields

\[ \mathbf{v}^T (\mathbf{A}^T \mathbf{A} - \mathbf{I}) \mathbf{v} = 0. \]

The only way this can be true of all vectors \( \mathbf{v} \) is if

\[ \mathbf{A}^T \mathbf{A} - \mathbf{I} = 0, \]

or

\[ \mathbf{A}^T \mathbf{A} = \mathbf{I}. \]

**Q4:**

\[ \det(\mathbf{A}) = \det(\mathbf{UWV}^T) = \det(\mathbf{U}) \det(\mathbf{W}) \det(\mathbf{V}^T). \]

Since \( \mathbf{U} \) and \( \mathbf{V} \) are orthonormal, their determinants are either 1 or -1. Hence, \( \det(\mathbf{A}) = 0 \) iff \( \det(\mathbf{W}) = 0 \). Also, it is easy to show that

\[ \det(\mathbf{W}) = \prod_{i=1}^{n} w_i, \]

where \( w_i \) are the singular values. Clearly, this product is 0 iff at least one of the singular values is 0. This completes the proof.

**Q7:** Evaluate the expression in two ways:

\[ \mathbf{v}_1^T \mathbf{A} \mathbf{v}_2 = \mathbf{v}_1^T \lambda_2 \mathbf{v}_2 = \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 \]

and

\[ \mathbf{v}_1^T \mathbf{A} \mathbf{v}_2 = \mathbf{v}_1^T \mathbf{A}^T \mathbf{v}_2 = (\mathbf{A} \mathbf{v}_1)^T \mathbf{v}_2 = \lambda_1 \mathbf{v}_1^T \mathbf{v}_2 \]

Equating these, we have

\[ \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 = \lambda_1 \mathbf{v}_1^T \mathbf{v}_2. \]

Since \( \lambda_1 \neq \lambda_2 \), the only way these can be equal is if \( \mathbf{v}_1^T \mathbf{v}_2 = 0 \). This means, of course, that \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are orthogonal.