FD VII: Fuzzy AHP

Weighted min *and* max *aggregations*

- R.R.Yager, On weighted median aggregation, *International Journal of Uncertainty, Fuzziness and Knowledge-based Systems*, 1(1994) 101-113.

In many applications of fuzzy sets as multi-criteria decision making, pattern recognition, diagnosis and fuzzy logic control one faces the problem of weighted aggregation.

Assume associated with each fuzzy set A_i (characterizing the i-th attribute or criterion) is a weight $w_i \in [0, 1]$ indicating its importance in the aggregation procedure, $i = 1, \dots, n$.

The general process for the inclusion of importance in the aggregation involves the transformation of the fuzzy sets under the importance.

Let **Agg** indicate an aggregation operator, max or min, to find the weighted aggregation.

Let x be an alternative and let

$$(a_1, a_2, \dots, a_n)$$

denote the degrees to which x satisfies the criteria, i.e.

$$a_i = A_i(x), \ i = 1, \dots, n.$$

We first transform each of the membership grades using the weights

$$g(w_i, a_i) = \hat{a}_i, \ i = 1, \dots, n$$

and then obtain the weighted aggregate

$$A(x) = \mathbf{Agg}\langle \hat{a}_1, \ldots, \hat{a}_n \rangle.$$

The form of g depends upon the type of aggregation being performed, the operation **Agg**.

As discussed by Yager in incorporating the effect of the importances in the min operation we are interested in reducing the effect of the elements which have low importance.

In performing the min aggregation it is the elements with low values that play the most significant role in this type of aggregation, one way to reduce the effect of elements with low importance is to transform them into big values, values closer to one.

Yager introduced a class of functions which can be used for the inclusion of importances in the min aggregation

$$g(w_i, a_i) = S(1 - w_i, a_i)$$

where S is a t-conorm. Three significant manifestations of this operation are

$$g(w_i, a_i) = \max\{1 - w_i, a_i\} \text{ (MAX t-conorm)}$$

$$g(w_i, a_i) = 1 - w_i + a_i w_i \ (S(a, b) = a + b - ab)$$

$$g(w_i, a_i) = \min\{1, 1 - w_i + a_i\} \ (S(a, b) = \min\{1, a + b\})$$

We first note that if $w_i = 0$ then from the basic property of t-conorms it follows that

$$S(1 - w_i, a_i) = S(1, w_i) = 1$$

Thus, zero importance gives us one.

Example

Let

$$w = (0.3, 0.2, 0.1, 0.4, 0.0)$$

be the vector of weights and let

$$(0.4, 0.6, 0.6, 0.4, 0.8)$$

be the vector of aggregates.

- If $g(w_i, a_i) = \max\{1 - w_i, a_i\}$ then we get

$$g(w_1, a_1) = g(0.3, 0.4) = (1 - 0.3) \vee 0.4 = 0.7$$
$$g(w_2, a_2) = g(0.2, 0.6) = (1 - 0.2) \vee 0.6 = 0.8$$
$$g(w_3, a_3) = g(0.1, 0.6) = (1 - 0.1) \vee 0.6 = 0.9$$
$$g(w_4, a_4) = g(0.4, 0.4) = (1 - 0.4) \vee 0.4 = 0.6$$
$$g(w_5, a_5) = g(0.0, 0.8) = (1 - 0.0) \vee 0.8 = 1.0$$

  That is

$$\min\{\hat{a}_1, \dots, \hat{a}_n\} = \min\{0.7, 0.8, 0.9, 0.6, 1\} = 0.6$$

- If $g(w_i, a_i) = 1 - w_i + a_i w_i$ then we get

$$g(w_1, a_1) = g(0.3, 0.4) = (1 - 0.3) + 0.4 \times 0.3 = 0.82$$
$$g(w_2, a_2) = g(0.2, 0.6) = (1 - 0.2) + 0.6 \times 0.2 = 0.92$$
$$g(w_3, a_3) = g(0.1, 0.6) = (1 - 0.1) + 0.6 \times 0.1 = 0.96$$
$$g(w_4, a_4) = g(0.4, 0.4) = (1 - 0.4) + 0.4 \times 0.4 = 0.76$$
$$g(w_5, a_5) = g(0.0, 0.8) = (1 - 0.0) + 0.8 \times 0.0 = 1.0$$

  That is

$$\min\{\hat{a}_1, \dots, \hat{a}_n\} = \min\{0.82, 0.92, 0.96, 0.76, 1\} = 0.76$$

As for the max aggregation operator: Since it is the large values that play the most important role in the aggregation we desire to transform the low importance elements into small values and thus have them not play a significant role in the max aggregation.

Yager suggested a class of functions which can be used for importance transformation in max aggregation

$$g(w_i, a_i) = T(w_i, a_i)$$

where T is a t-norm.

The three corresponding manifestations are

$$h(w_i, a_i) = \min\{w_i, a_i\} \text{ (minimum)}$$

$$h(w_i, a_i) = w_i a_i \text{ (probabilistic)}$$

$$h(w_i, a_i) = 0 \vee (w_i + a_i - 1) \text{ (Łukasiewicz)}$$

We see that if $w_i = 0$ then $T(w_i, a_i) = 0$ and the element plays no rule in the max. Furthermore, as w_i decreases then $T(w_i, a_i)$ decreases.

Example

Let

$$w = (0.3, 0.2, 0.1, 0.4, 0.0)$$

be the vector of weights and let

$$(0.4, 0.6, 0.6, 0.4, 0.8)$$

be the vector of aggregates.

- If $g(w_i, a_i) = \min\{w_i, a_i\}$ then we get

$$g(w_1, a_1) = g(0.3, 0.4) = 0.3 \wedge 0.4 = 0.3$$

$$g(w_2, a_2) = g(0.2, 0.6) = 0.2 \wedge 0.6 = 0.2$$
$$g(w_3, a_3) = g(0.1, 0.6) = 0.1 \wedge 0.6 = 0.1$$
$$g(w_4, a_4) = g(0.4, 0.4) = 0.4 \wedge 0.4 = 0.6$$
$$g(w_5, a_5) = g(0.0, 0.8) = 0.0 \wedge 0.8 = 0.0$$

That is

$$\max\{\hat{a}_1, \dots, \hat{a}_n\} = \min\{0.3, 0.2, 0.1, 0.6, 0.0\} = 0.6$$

- If $g(w_i, a_i) = w_i a_i$ then we get

$$g(w_1, a_1) = g(0.3, 0.4) = 0.4 \times 0.3 = 0.12$$
$$g(w_2, a_2) = g(0.2, 0.6) = 0.6 \times 0.2 = 0.12$$
$$g(w_3, a_3) = g(0.1, 0.6) = 0.6 \times 0.1 = 0.06$$
$$g(w_4, a_4) = g(0.4, 0.4) = 0.4 \times 0.4 = 0.16$$
$$g(w_5, a_5) = g(0.0, 0.8) = 0.8 \times 0.0 = 0.0$$

That is

$$\max\{\hat{a}_1, \dots, \hat{a}_n\} = \max\{0.12, 0.12, 0.06, 0.16, 0.0\} = 0.16$$

Assume associated with each fuzzy set A_i is a weight $w_i \in [0, 1]$ indicating its importance in the aggregation procedure, $i = 1, \dots, n$.

Let **Agg** indicate any aggregation operator, let x be an alternative and let

$$(a_1, a_2, \dots, a_n)$$

denote the degrees to which x satisfies the criteria. We define the weighted aggregation as

$$A(x) = \mathbf{Agg}\langle \hat{a}_1, \dots, \hat{a}_n \rangle.$$

where

$$g(w_i, a_i) = \hat{a}_i,\ i = 1, \dots, n$$

where the function g satisfies the following properties

- if $a > b$ then $g(w,a) \geq g(w,b)$
- $g(w,a)$ is monotone in w
- $g(0,a) = id$
- $g(1,a) = a$

where the identity element, id, is such that if we add it to our aggregates it doesn't change the aggregated value.

The importances of the criteria could be approximated by the

Analytic Hierarchy Process (AHP)

using pairwise comparisons.

- T.L.Saaty, *The Analytic Hierarchy Process: Planning, Priority Setting, Resource Allocation* (McGraw-Hill, New York, 1980).

Assume we have p objectives and we want to construct a scale, rating these objectives as to their importance with respect to the decision, as seen by the analyst.

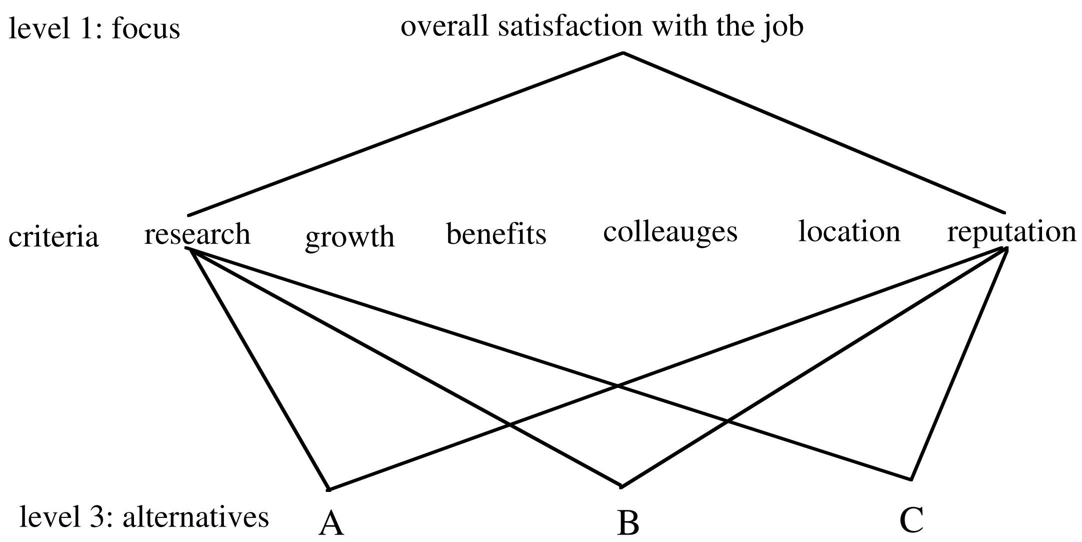
We ask the decision maker to compare the objectives in paired comparisons. If we are comparing objective i with objective j, we assign the values a_{ij} and a_{ji} as follows:

- $$a_{ij} = \frac{1}{a_{ji}}$$
- If objective i is more important than objective j then a_{ij} gets assigned a number as follows:

Intensity of relative importance	Definition
1	equal importance
3	weak importance (of one over the other)
5	strong importance
7	demonstrated importance over the other
9	absolute importance
2,4,6,8	intermediate values between

Saaty's scale of relative importances.

We illustrate Saaty's method on a job selection problem (3 alternatives compared on 6 criteria)



Choice of job.

The question asked was, which of a given pair of criteria is seen as contributing more to overall satisfaction with a job and what is the intensity or strength of the difference?

	res.	growth	benefits	coll.	location	reputation	priority
research	1	1	1	4	1	1/2	0.16
growth	1	1	2	4	1	1/2	0.19
benefits	1	1/2	1	5	3	1/2	0.19
colleaug.	1/4	1/4	1/5	1	1/3	1/3	0.05
location	1	1	1/3	3	1	1	0.12
reputation	2	2	2	3	1	1	0.30

Pairwise comparison matrix of criteria.

The relative weights of criteria (priorities) can be computed as normalized geometric means of the rows (which are very close to the eigenvector corresponding to the largest eigenvalue of the matrix)

The geometric means are computed as

$$m_1 = \sqrt[6]{1 \times 1 \times 1 \times 4 \times 1 \times 1/2}$$

$$m_2 = \sqrt[6]{1 \times 1 \times 2 \times 4 \times 1 \times 1/2}$$

$$m_3 = \sqrt[6]{1 \times 1/2 \times 1 \times 5 \times 3 \times 1/2}$$

$$m_4 = \sqrt[6]{1/4 \times 1/4 \times 1/5 \times 1 \times 1/3 \times 1/3}$$

$$m_5 = \sqrt[6]{1 \times 1 \times 1/3 \times 3 \times 1 \times 1}$$

$$m_6 = \sqrt[6]{2 \times 2 \times 2 \times 3 \times 1 \times 1}$$

So, the relative weight (priority) of the criterion *research* is obtained as

$$p_1 = \frac{m_1}{m_1 + m_2 + m_3 + m_4 + m_5 + m_6}$$

Then we compare the alternatives on each of the criteria

research	A	B	C	priority
A	1	1/4	1/2	0.14
B	4	1	3	0.63
C	2	1/3	1	0.24

growth	A	B	C	priority
A	1	1/4	1/2	0.10
B	4	1	3	0.33
C	2	1/3	1	0.57

benefits	A	B	C	priority
A	1	3	1/3	
B	1/3	1		
C	3	1	1	

colleaug.	A	B	C	priority
A	1	1/3	5	
B	3	1	7	
C	1/5	1/7	1	

location	A	B	C	priority
A	1	1	7	
B	1	1	7	
C	1/7	1/7	1	

reputat.	A	B	C	priority
A	1	7	9	0.77
B	1/7	1	5	0.17
C	1/9	1/5	1	0.05

We obtain

$$0.16 \times \begin{pmatrix} 0.14 \\ 0.63 \\ 0.24 \end{pmatrix} + 0.19 \times \begin{pmatrix} 0.10 \\ 0.33 \\ 0.57 \end{pmatrix} + \cdots + 0.30 \times \begin{pmatrix} 0.77 \\ 0.17 \\ 0.05 \end{pmatrix} = \begin{pmatrix} 0.40 \\ 0.34 \\ 0.26 \end{pmatrix} \quad \begin{matrix} A \\ B \\ C \end{matrix}$$

So job A should be selected as the best alternative.

Given a decision problem

$$\begin{array}{ccccc} & C_1 & C_2 & \dots & C_k \\ A_1 & a_{11} & a_{12} & \dots & a_{1k} \\ A_2 & a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \vdots & & \vdots \\ A_n & a_{n1} & a_{n2} & \dots & a_{nk} \end{array}$$

Yager

- R.R.Yager, Fuzzy decision making using unequal objectives, *Fuzzy Sets and Systems*, 1(1978) 87-95.

computes the relative importance for each criteria by Saaty's method and then the unit score of the i-th alternative is computed as

$$\min \{\hat{a}_{i1}, \dots, \hat{a}_{ik}\}$$

where

$$\hat{a}_{ij} = a_{ij}^{p_j},$$

and p_j denotes the priority of the j-th criterion $j = 1, \dots, k$.

The membership grade in all objectives having little importance $p \to 0$ becomes larger.

Example Suppose that the vector of relative importances is computed as

$$p = (0.16, 0.19, 0.19, 0.05, 0.12, 0.30)$$

and the ratings are

$$\begin{array}{ll} A & (1.0, 0.6, 1.0, 0.5, 1, 0.2) \\ B & (0.2, 1, 0.7, 0.5, 0.8, 1.0) \\ C & (0.8, 1, 0.4, 0.6, 0.2, 1.0) \end{array}$$

Then the unit scores are

$$\begin{array}{ll} A & \min\{1.0,\ 0.6^{0.19},\ 1.0,\ 0.5^{0.05},\ 1,\ 0.2^{0.30}\} \\ B & \min\{0.2^{0.16},\ 1,\ 0.7^{0.19},\ 0.5^{0.05},\ 0.8^{0.12},\ 1\} \\ C & \min\{0.8^{0.16},\ 1,\ 0.4^{0.19},\ 0.6^{0.05},\ 0.2^{0.12},\ 1\} \end{array}$$

That is

$$\begin{array}{ll} A & \min\{0.90, 0.96, 0.61\} = 0.61 \\ B & \min\{0.77, 0.93, 0.96, 0.97\} = 0.77 \\ C & \min\{0.96, 0.84, 0.97, 0.82\} = 0.82 \end{array}$$

So job C is selected.