# FD V: Fuzzy approaches to MOP

Consider a multiple objective program

$$\max_{x \in X} \{f_1(x), \dots, f_k(x)\}$$

where

- $f_i : \mathcal{R}^n \to \mathcal{R},\ i = 1, \dots, k$ are objective functions, $\mathcal{R}^k$ is the *criterion space*,
- $x = (x_1, \dots, x_n)^T \in \mathcal{R}^n$ is the decision variable, $\mathcal{R}^n$ is the *decision space*,
- $X$ is a subset of $\mathcal{R}^n$ without any additional conditions for the moment. $X$ is called the *set of feasible alternatives.*
- The image of $X$ in $\mathcal{R}^k$, denoted by $Z_X$, i.e. the set of *feasible outcomes* is defined as

  $$Z_X = \{z \in \mathcal{R}^k | z_i = f_i(x), i = 1, \dots, k, x \in X\}$$

An $x^* \in X$ is said to be efficient (or nondominated or Pareto-optimal) for the MOP iff there exists no $y \in X$ such that

$$f_i(y) \geq f_i(x^*)$$

for all $i$ with strict inequality for at least one $i$. The set of all Pareto-optimal solutions will be denoted by $X^*$.

In other words, a nondominated point is such that any other point in $Z_X$ which increases the value of one criterion also decreases the value of at least one other criterion.

Suppose that we are given a three-objective decison problem

$$\max_{x \in X} \{f_1(x), f_2(x), f_3(x)\},$$

where $X = \{u, y, z\}$ is a finite set and let

$$\{f_1(u), f_2(u), f_3(u)\} = (1, 3, 3),$$

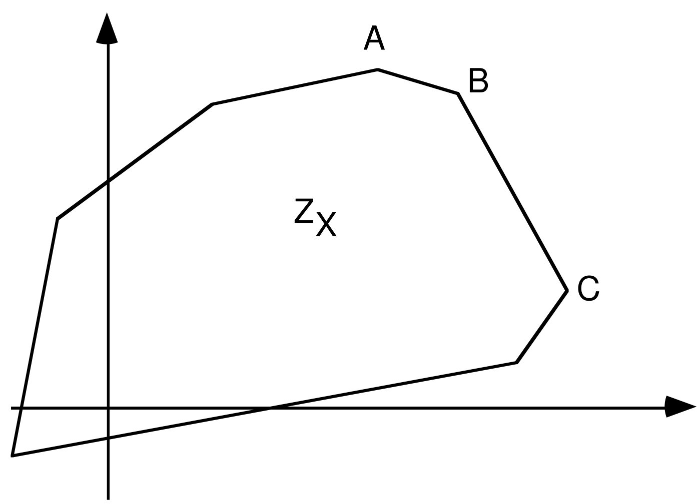
$$\{f_1(y), f_2(y), f_3(y)\} = (2, 2, 3)$$

$$\{f_1(z), f_2(z), f_3(z)\} = (1, 2, 2)$$

be the set of feasible outcomes, i.e.

$$Z_X = \{(1, 3, 3), (2, 2, 3), (1, 2, 2)\}$$

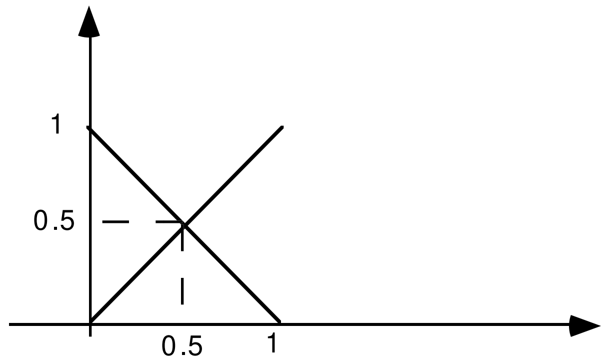
Then $u$ and $y$ are the effecient solutions.




Segments $AB$ and $BC$ are Pareto-optimal

A typical biobjective MOP is the following

$$\max\{x, 1-x\}$$
$$x \in [0,1]$$




Here we have $X^* = [0,1]$, i.e. each feasible alternative is an efficient solution.

A natural way to obtain initial information about the decision problem is to optimize each criterion separately over $X$. Let $x^i$ be a solution to

$$\max\{f_i(x)|x \in X\}$$

and let $M_i = f_i(x^i)$ be the optimal value of the $i$-th individual objective function over $X$.

The vector $M = (M_1, M_2, \ldots, M_k)$ is called the *ideal point*. The ideal point for the biobjective problem is $(1,1)$ (which is not attainable by any feasible point).

The *payoff matrix* of an $k$-objective problem is defined as

$$
\begin{array}{ccccc}
 & x^1 & x^2 & \ldots & x^k \\
f_1 & M_1 & f_1(x^2) & \ldots & f_1(x^k) \\
f_2 & f_2(x^1) & M_2 & \ldots & f_2(x^k) \\
\vdots & \vdots & \vdots & & \vdots \\
f_k & f_k(x^1) & f_k(x^2) & \ldots & M_k
\end{array}
$$

Here we hace used the equation $M_i = f_i(x^i)$.

The payoff matrix for the biobjective function is

$$
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

Let $m_i$ denote the value

$$
\min\{f_i(x)|x \in X\}
$$

i.e. $m_i$ is the worst possible value for the $i$-th objective.

It is clear that the inequalities

$$
m_i \leq f_i(x) \leq M_i
$$

hold for each $x$ in $X$.

In the case of the biobjective problem we have

$$
0 \leq f_1(x) = x \leq 1, \; 0 \leq f_2(x) = 1 - x \leq 1
$$

for all $x$ from $X = [0, 1]$.

# Application functions for MOP

An application function $h_i$ for the MOP

$$\max_{x \in X} \{f_1(x), \dots, f_k(x)\}$$

is defined as

$$h_i : \mathcal{R} \to [0, 1]$$

such that $h_i(t)$ measures the degree of fulfillment of the decision maker's requirements about the $i$-th objective by the value $t$.

- M.Delgado,J.L.Verdegay and M.A.Vila, A possibilistic approach for multuobjective programming problems. Efficiency of solutions, in: R.Slowinski and J.Teghem eds., *Stochastic versus Fuzzy Approaches to Multiobjective Mathematical Programming under Uncertainty*, Kluwer Academic Publisher, Dordrecht, 1990 229-248.

Suppose that the decision maker has some preference parameters, for example

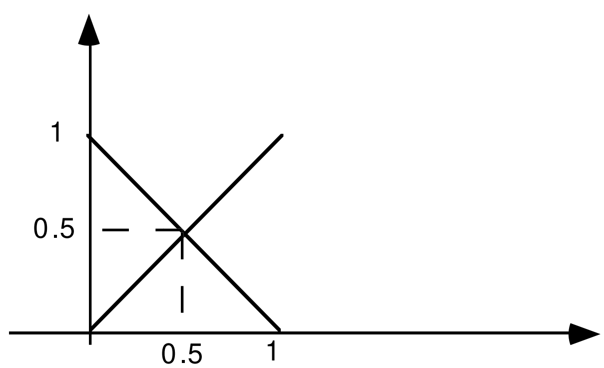
- *reference points* which represents desirable leveles on each criterion
- *reservation levels* which represent minimal requirements on each criterion

If the value of an objective function (at the current point) exceeds his desirable level on this objective then he is totally satisfied with this alternative.

If, however, he value of an objective function (at the current point) is below of his reservation level on this objective then he is absolutely not satisfied with this alternative.

Consider again the biobjective problem

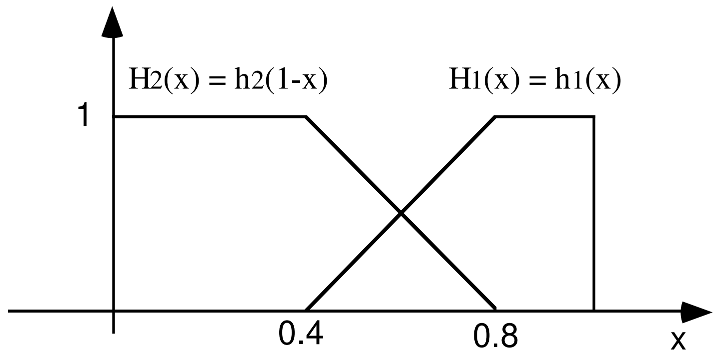
$$\max\{x, 1 - x\}$$
$$x \in [0, 1]$$




For example, we can introduce the following linear application functions

$$h_1(t) = h_2(t) = \begin{cases} 1 & \text{if } t \geq 0.8 \\ 1 - \frac{|t-0.4|}{0.4} & \text{if } 0.4 \leq t \leq 0.8 \\ 0 & \text{if } t \leq 0.4 \end{cases}$$

where $t$ denotes the value attained by the objective functions.




These application functions can be interpreted as:

If I can find a feasible alternative where the values of both objectives exceeds 0.8 then I am totally satisfied with this solution.

On the other hand, alternatives for which the values of both objectives are less than 0.4 are not qualified candidates for "good solutions".

Here

$$H_1(x) = h_1(f_1(x)) = h_1(x)$$

and

$$H_2(x) = h_2(f_2(x)) = h_2(1 - x).$$

In other words, with the notation of

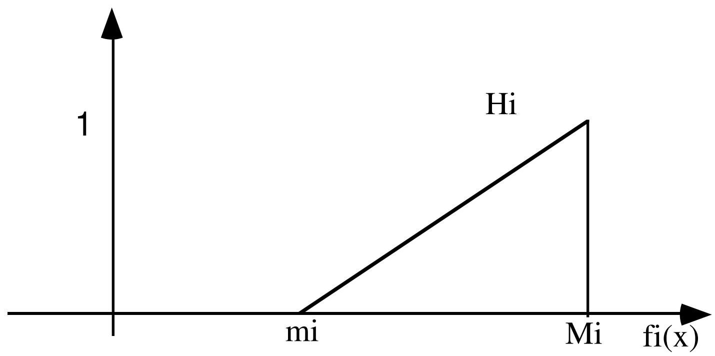
$$H_i(x) = h_i(f(x)),$$

$H_i(x)$ may be considered as the degree of membership of $x$ in the fuzzy set "good solutions" for the $i$-th objective.

A generally used application function is the following

$$H_i(x) = h_i(f_i(x)) = 1 - \frac{M_i - f_i(x)}{M_i - m_i}$$

where $M_i$ denotes the independent maximum ad $m_i$ stands for the independent minimum of the $i$-th objective function over $X$.




It is clear that if for some alternative $x^*$,

$$f_i(x^*) = M_i$$

(ideal solution for the $i$-th objective) then

$$H_i(x^*) = 1,$$

and if for some alternative $x_*$,

$$f_i(x_*) = m_i$$

(anti-ideal solution for the $i$-th objective) then

$$H_i(x_*) = 0.$$

The bigger the value of the objective the bigger the satisfaction of the decision maker.

Then a

"good compromise solution"

to MOP may be defined as an $x \in X$ being

"as good as possible"

for the whole set of objectives.

Taking into consideration the nature of $H_i$, it is quite reasonable to look for such a kind of solution by means of the following auxiliary problem

$$\max_{x \in X} \{H_1(x), \ldots, H_k(x)\}$$

For

$$\max\{H_1(x), \ldots, H_k(x)\}$$

may be interpreted as a synthetical notation of a conjuction statement

*maximize jointly all objectives*

and $H_i(x) \in [0, 1]$, it is reasonable to use a t-norm $T$ to represent the connective AND.

In this way

$$\max_{x \in X} \{H_1(x), \ldots, H_k(x)\}$$

turns into the single-objective problem

$$\max_{x \in X} T(H_1(x), \ldots, H_k(x)).$$

There exist several ways to introduce application functions. Usually, the authors consider increasing membership functions of the form

$$h_i(t) = \begin{cases} 1 & \text{if } t \geq R_i \\ v_i(t) & \text{if } m_i \leq t \leq R_i \\ 0 & \text{if } t \leq r_i \end{cases}$$

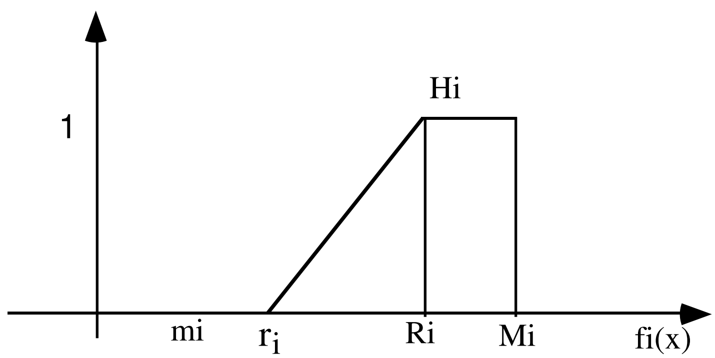
where $r_i$

$$r_i \geq \min\{f_i(x)|x \in X\}$$

denotes the reservation level representing minimal requirement and $R_i$

$$R_i \leq \max\{f_i(x)|x \in X\}$$

is the desirable level on the $i$-th objective.




Linear membership function.

Consider again the biobjective problem

$$\max\{x, 1-x\}$$

$$x \in [0, 1]$$

Using the general application functions

$$H_1(x) = 1 - \frac{1-x}{1}, \quad H_2(x) = 1 - \frac{1-(1-x)}{1}$$

and choosing the minimum-norm to aggregate the values of objective functions the resulting single objective problem

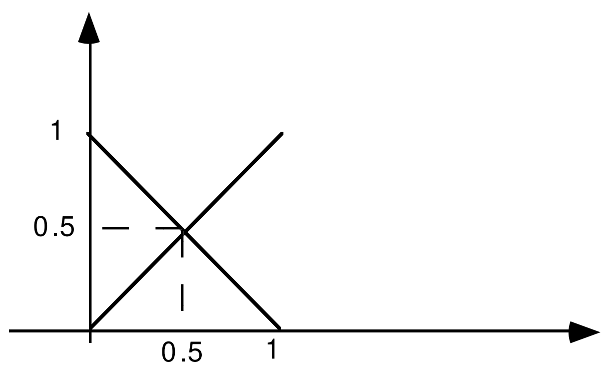
$$\max \min\{x, 1-x\}$$

$$x \in [0,1]$$

has a unique solution

$$x^* = 1/2$$

and the optimal values of the objective functions are $(0.500, 0.500)$.




If, however we use the Łukasiewicz t-norm for criteria aggregation then the solution set of single objective problem

$$\max \max\{x + 1 - x, 0\}$$

$$x \in [0,1]$$

is the whole unit interval $[0,1]$.

Alternatively, we might have the following linear application functions

$$h_1(t) = \begin{cases} 1 & \text{if } t \geq 0.8 \\ 1 - \frac{|t-0.4|}{0.4} & \text{if } 0.4 \leq t \leq 0.8 \\ 0 & \text{if } t \leq 0.4 \end{cases}$$

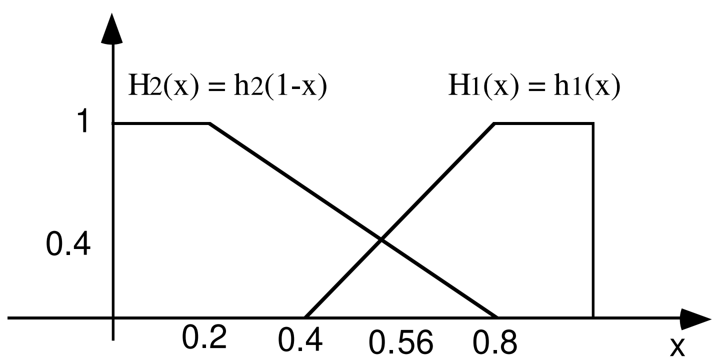
$$h_2(t) = \begin{cases} 1 & \text{if } t \geq 0.8 \\ 1 - \frac{|t-0.2|}{0.6} & \text{if } 0.2 \leq t \leq 0.8 \\ 0 & \text{if } t \leq 0.2 \end{cases}$$

where $t$ denotes the value attained by the objective functions.

Choosing the minimum norm for aggregation the problem

$$\max \min\{H_1(x), H_2(x)\}$$
$$x \in [0, 1]$$

has a unique solution $x^* = 0.56$ and the optimal values of the objectives are $(0.4, 0.4)$.




One of the most important questions is the efficiency of the obtained compromise solutions.

**Theorem** Let $x^*$ be an optimal solution to

$$\max_{x \in X} T(H_1(x), \ldots, H_k(x))$$

where

- $T$ is a t-norm,
- $H_i(x) = h_i(f_i(x))$, $h_i$ is an *increasing* application function, $i = 1, \ldots, k$.

Let $h_i$ be strictly increasing on the interval

$$[r_i, R_i]$$

for $i = 1, \ldots, k$. Then $x^*$ is efficient for the problem

$$\max_{x \in X} \{f_1(x), \ldots, f_k(x)\}$$

if either

(i) $x^*$ is unique;

(ii) T is strict and $H_i(x^*) = h_i(f_i(x^*)) \in (0, 1)$ for $i = 1, \ldots, k$.

**Proof** (i) Suppose that $x^*$ is not efficient. If $x^*$ were dominated, then $x^{**} \in X$ such that

$$f_i(x^*) \leq f_i(x^{**})$$

for all $i$ and with a strict inequality for at least one $i$.

Consequently, from the monotonicity of $T$ and $h_i$ we get

$$T(H_1(x^*), \ldots, H_k(x^*)) \leq T(H_1(x^{**}), \ldots, H_k(x^{**}))$$

which means that $x^{**}$ is also an optimal solution to the auxiliary problem. So $x^*$ is not unique.

(ii) Suppose that $x^*$ is not efficient. If $x^*$ were dominated, then $x^{**} \in X$ such that

$$f_i(x^*) \leq f_i(x^{**})$$

for all $i$ and with a strict inequality for at least one $i$.

Taking into consideration that

$$H_i(x^*) = h_i(f_i(x^*)) \in (0, 1)$$

for all $i$ and $T$ is strict, and $h_i$ is monoton increasing we get

$$T(H_1(x^*), \ldots, H_k(x^*)) < T(H_1(x^{**}), \ldots, H_k(x^{**}))$$

which means that $x^*$ is not an optimal solution to the auxiliary problem. So $x^*$ is not efficient.

If we use linear application functions then they are strictly increasing on $[r_i, R_i]$, and, therefore any optimal solution $x^*$ to the auxiliary problem is an efficient solution to the original MOP problem if either (i) $x^*$ is unique; (ii) T is strict and $H_i(x^*) \in (0, 1),\ i = 1, \ldots k$.