FD IV: Fuzzy Linear Programming

Linear programming with flexible constraints

- H.-J. Zimmermann Description and optimization of fuzzy systems, *Internat. J. General Systems* **2** (1975) 209-215.

The conventional model of Linear Programming (LP) can be stated as

$$\langle c, x\rangle \to \min$$
$$\text{s. t. } Ax \le b.$$

where

- $\langle c, x\rangle = c_1x_1 + \cdots + c_nx_n$ is called the objective function,
- $x = (x_1, \dots, x_n)^T$ is the decision variable,
- $(Ax)_i = \langle a_i, x\rangle = a_{i1}x_1 + \cdots + a_{in}x_n$
- $X = \{x|Ax \le b\}$ is the set of *feasible* alternatives.
- x_{opt} is called a solution to LP problem if

  $$\langle c, x_{opt}\rangle \le \langle c, x\rangle$$

  for all $x \in X$.

In many real-world problems instead of minimization of the objective function it may be sufficient to determine an x such that

$$
\begin{aligned}
c_1 x_1 + \cdots + c_n x_n &\leq b_0, \\
a_{11} x_1 + \cdots + a_{1n} x_n &\leq b_1, \\
&\vdots \\
a_{m1} x_1 + \cdots + a_{mn} x_n &\leq b_m,
\end{aligned}
$$

where b_0 is a predetermined *aspiration level*.

The fuzzy objective function is characterized by its membership function, and so are the constraints.

Since we want to satisfy (optimize) the objective function as well the constraints, a decision in a fuzzy environment is defined in analogy to nonfuzzy environment as the selection of activities which simultaneously satisfy objective function(s) *and* constraints.

In fuzzy set theory the intersection of sets normally corresponds to the logical *and*.

The decision in fuzzy environment can therefore be viewed as the intersection of fuzzy constraints and fuzzy objective function(s).

The relationship between constraints and objective functions in a fuzzy environment is therefore fully symmetric, i.e. there is no longer difference between the former and latter.

Starting from the problem

$$\langle c, x\rangle \to \min$$
$$\text{s. t. } Ax \le b.$$

the adopted fuzzy version is

$$\langle c, x\rangle \lesssim b_0$$
$$\text{s. t. } Ax \lesssim b.$$

That is

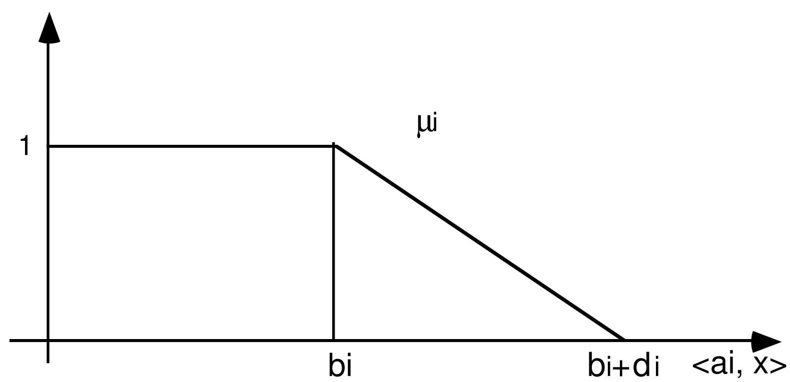
$$c_1x_1 + \cdots + c_nx_n \lesssim b_0,$$
$$a_{11}x_1 + \cdots + a_{1n}x_n \lesssim b_1,$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n \lesssim b_m,$$

Here b_0 means an aspiration level of the decision maker.

We now define the membership functions for the constraints

$$\mu_i(x) = \begin{cases} 1 & \text{if } \langle a_i, x\rangle \le b_i \\ 1 - \frac{\langle a_i, x\rangle - b_i}{d_i} & \text{if } b_i < \langle a_i, x\rangle \le b_i + d_i \\ 0 & \text{if } \langle a_i, x\rangle > b_i + d_i, \end{cases}$$

where $d_i > 0$ are subjectively chosen constants of admissible violations, $i = 1, \dots, m$.

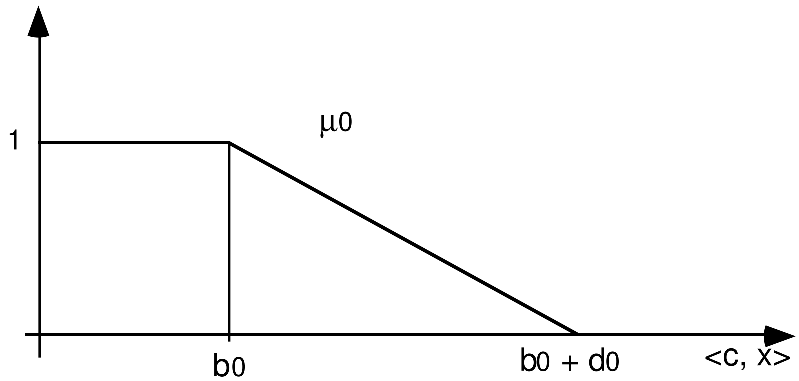


$\mu_i(x)$ denotes the degree to which x satisfies the ith constraint

We define the membership functions for objective function

$$\mu_0(x) = \begin{cases} 1 & \text{if } \langle c, x\rangle \le b_0 \\ 1 - \frac{\langle c, x\rangle - b_0}{d_0} & \text{if } b_i < \langle c, x\rangle \le b_0 + d_0 \\ 0 & \text{if } \langle c, x\rangle > b_0 + d_0, \end{cases}$$

where $d_0 > 0$ are subjectively chosen constants of admissible violations.



$\mu_0(x)$ denotes the degree to which x satisfies the fuzzy goal function

The *fuzzy decision* is defined by Bellman and Zadeh's principle as

$$D(x) = \min\{\mu_0(x), \mu_1(x), \ldots, \mu_m(x)\}$$

where x can be any element of the n dimensional space, because any element has a degree of feasibility, which is between zero and one.

And an optimal solution to the fuzzy LP is determined from the relationship

$$D(x_{opt}) = \max_x D(x).$$

How can we solve the resulting (crisp) problem

$$D(x) \rightarrow \max?$$

We should find an x which satisfies the constraints and the fuzzy objective function with the biggest possible degree.

$$\lambda \to \max$$
$$\min\{\mu_0(x), \mu_1(x), \dots, \mu_m(x)\} \ge \lambda$$
$$\lambda \in [0,1], x \in \Re^n$$

$$\lambda \to \max$$
$$\mu_0(x) \ge \lambda,\ \ \mu_1(x) \ge \lambda,\ \ \dots \mu_m(x) \ge \lambda$$
$$\lambda \in [0,1], x \in \Re^n$$

$$\lambda \to \max$$
$$1 - \frac{\langle c, x\rangle - b_0}{d_0} \ge \lambda, \quad 1 - \frac{\langle a_i, x\rangle - b_i}{d_i} \ge \lambda,$$
$$i = 1, \dots, m,\ \lambda \in [0,1], x \in \Re^n$$

The resulting crisp LP problem can be written as

$$\lambda \to \max$$
$$\langle c, x\rangle - b_0 \ge d_0(1-\lambda)$$
$$\langle a_i, x\rangle - b_i \ge d_i(1-\lambda),$$
$$i = 1, \dots, m,\ \lambda \in [0,1], x \in \Re^n$$

Let λ^* and x^* be an optimal solution to the above problem. Then x^* is the considered as a best compromise solution, and λ^* is interpreted as the degree of reality (or achievability) of x^*.