FD III: Aggregation operators

Let us look at the process for combining the individual ratings

$$(A_1(x), \ldots, A_n(x))$$

of an alternative x.

A basic assumption we shall make is that the operation is **pointwise** and **likewise**.

By pointwise we mean that for every alternative x, $A(x)$ just depends upon

$$A_j(x),\ j = 1, \ldots, n.$$

By likewise we mean that the process used to combine the A_j is the same for all of the x.

Let us denote the pointwise process we use to combine the individual ratings as

$$A(x) = \mathbf{Agg}\,[A_1(x), \ldots, A_n(x)]$$

In the above **Agg** is called the aggregation operator and the $A_j(x)$ are the arguments.

More generally, we can consider this as an operator

$$a = \mathbf{Agg}\,(a_1, \ldots, a_n)$$

where the a_i and a are values from the membership grade space, normally the unit interval.

Let us look at the minimal requirements associated with **Agg**.

We first note that the combination of of the individual ratings should be independent of the choice of indexing of the criteria.

This implies that a required property that we must associate with the Agg operator is that of commutativity, the indexing of the arguments does not matter.

We note that the commutativity property allows to represent the arguments of the **Agg** operator, as an unordered collection of possible duplicate values;

such an object is a **bag**.

For an individual rating, A_j, the membership grade $A_j(x)$ indicates the degree or strength to which this rate suggests that x is the appropriate (best compromise) solution.

In particular if for a pair of elements x_1 and x_2 it is the case that

$$A_j(x_1) \geq A_j(x_2),$$

then we are saying that the j-th rating is preferring x_1 as the output over x_2.

From this we can reasonably conclude that if all criterions prefer x_1 over x_2 as output then the overall rating should prefer x_1 over x_2.

This observation requires us to impose a monotonicity condition on the **Agg** operation.

In particular if

$$A_j(x_1) \geq A_j(x_2),$$

for all j, then

$$A(x_1) \geq A(x_2).$$

There appears one other condition we need to impose upon the aggregation operator.

Assume that there exists some criterion whose value does not matter.

The implication of this is that the criterion provides no information regarding what should be the overall rating.

It should not affect the final A.

The first observation we can make is that whatever rating this criterion provides should not make make any distinction between the potential overall ratings.

Thus, we see that the aggregation operator needs an *identy element*.

In summary, we see that the aggregation operator, **Agg** must satisfy three conditions:

- commutativity,
- monotonicity,
- must contain a fixed identity

These conditions are based on the three requirements: that the indexing of the criterions be unimportant, a positive association between individual criterion rating and total rating, and irrelevant criteria play no role in the decision process.

These operators are called

MICA

Monotonic Identity Commutative Aggregation

operators.

- R.R.Yager, Aggregation operators and fuzzy systems modeling, *Fuzzy Sets and Systems*, 67(1994) 129-145.

MICA *operators are the most general class for aggregation in fuzzy modeling.*

They include t-norms, t-conorms, averaging and compensatory operators.

Assume X is a set of elements. A bag drawn from X is any collection of elements which is contained in X.

A bag is different from a subset in that it allows multiple copies of the same element.

A bag is similar to a set in that the ordering of the elements in the bag does not matter.

If A is a bag consisiting of a, b, c, d we denote this as

$$A =< a, b, c, d > .$$

Assume A and B are two bags. We denote the sum of the bags

$$C = A \oplus B$$

where C is the bag consisting of the members of both A and B.

Example 1 *Let*

$$A =< a, b, c, d >$$

and

$$B =< b, c, c >$$

then

$$A \oplus B =< a, b, c, d, b, c, c >$$

In the following we let Bag(X) indicate the set of all bags of the set X.

Definition 1 *A function*

$$F\colon Bag(X) \rightarrow X$$

is called a bag mapping from $Bag(X)$ into the set X.

An important property of bag mappings are that they are commutative in the sense that the ordering of the elements does not matter.

Definition 2 *Assume*

$$A =< a_1, \ldots, a_n >$$

and

$$B =< b_1, \ldots, b_n >$$

are two bags of the same cardinality n.

If the elements in A and B can be indexed in such way that

$$a_i \geq b_i$$

for all i then we shall denote this

$$A \geq B.$$

Definition 3 *A bag mapping*

$$M\colon Bag([0,1]) \rightarrow [0,1]$$

is called MICA operator if it has the following two properties

- *If* $A \geq B$ *then* $M(A) \geq M(B)$
- *For every bag* A *there exists an element,* $u \in [0,1]$, *called the identity of* A *such that if*

  $$C = A \oplus < u >$$

  then

  $$M(C) = M(A)$$

Thus the MICA operator is endowed with two properties in addition to the inherent commutativity of the bag operator, *monotonicity and identity*:

- The requirement of monotonicity appears natural for an aggregation operator in that it provides some connection between the arguments and the aggregated value.
- The property of identity allows us to have the facility for aggregating data which does not affect the overall result. This becomes useful for enabling us to include importances among other characteristics.