FD I: Fuzzy Sets and Logic

Fuzzy sets were introduced by Zadeh as a means of representing and manipulating data that was not precise, but rather fuzzy.

- L.A.Zadeh, Fuzzy Sets, *Information and Control*, 8(1965) 338-353.

Definition 1 *(fuzzy set) Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function*

$$A \colon X \to [0, 1]$$

and $A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.

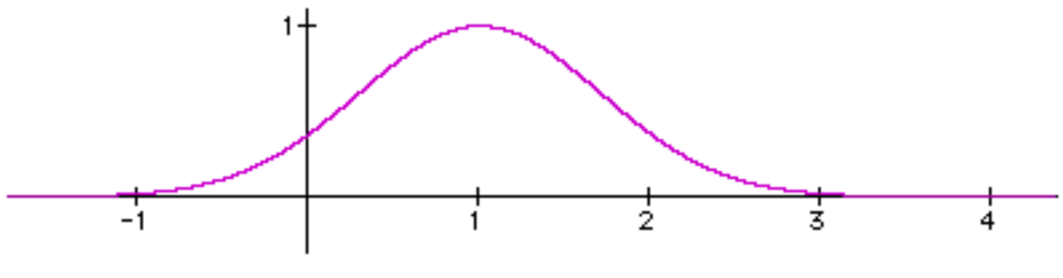
The value zero is used to represent complete non-membership, the value one is used to represent complete membership, and values in between are used to represent intermediate degrees of membership.

The mapping A is also called the membership function of fuzzy set A.

Example 1 *The membership function of the fuzzy set of* real *numbers "close to 1", is can be defined as*

$$A(t) = \exp(-\beta(t-1)^2)$$

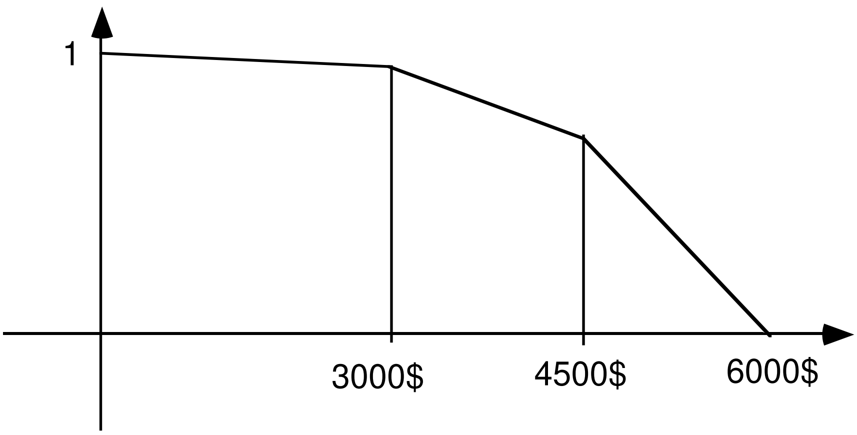
where β is a positive real number.



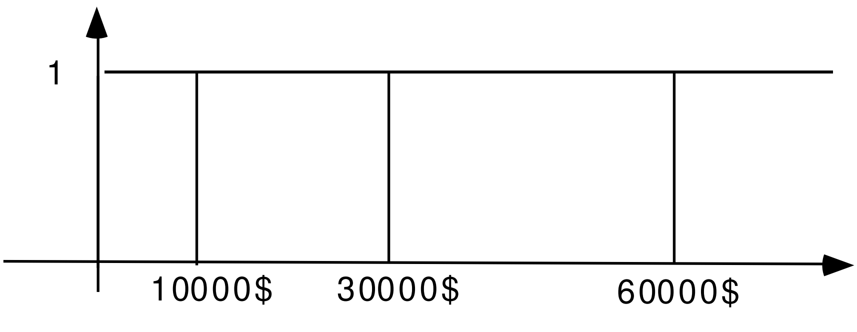
A membership function for "x is close to 1".

Example 2 *Assume someone wants to buy a cheap car.* Cheap *can be represented as a fuzzy set on a universe of prices, and depends on his purse. For instance, from the Figure* cheap *is roughly interpreted as follows:*

- *Below 3000$ cars are considered as cheap, and prices make no real difference to buyer's eyes.*
- *Between 3000$ and 4500$, a variation in the price induces a weak preference in favor of the cheapest car.*
- *Between 4500$ and 6000$, a small variation in the price induces a clear preference in favor of the cheapest car.*
- *Beyond 6000$ the costs are too high (out of consideration).*



Membership function of "cheap".

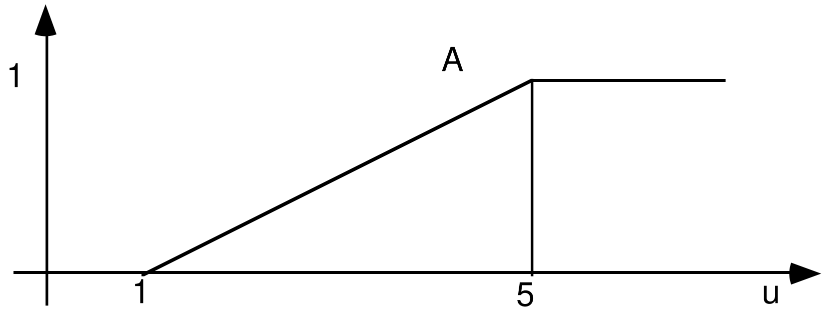


Price does not matter.

The membership function of the fuzzy set A, *big pressure*, can be interpreted as (see figure)

- 1 is in the fuzzy set *big pressure* with grade of membership 0
- 2 is in the fuzzy set *big pressure* with grade of membership 0.25
- 4 is in the fuzzy set *big pressure* with grade of membership 0.75
- x is in the fuzzy set *big pressure* with grade of membership 1, $x \geq 5$

$$A(u) = \begin{cases} 1 & \text{if } u \geq 5 \\ 1 - \frac{5-u}{4} & \text{if } 1 \leq u \leq 5 \\ 0 & \text{otherwise} \end{cases}$$



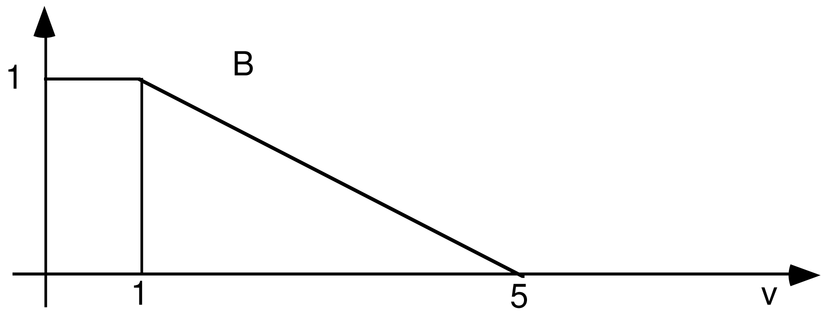
Membership function for "big pressure".

The membership function of the fuzzy set B, *small volume*, can be interpreted as (see figure)

- 5 is in the fuzzy set *small volume* with grade of membership 0
- 4 is in the fuzzy set *small volume* with grade of membership 0.25

- 2 is in the fuzzy set *small volume* with grade of membership 0.75
- x is in the fuzzy set *small volume* with grade of membership 1, $x \leq 1$

$$B(v) = \begin{cases} 1 & \text{if } v \leq 1 \\ 1 - \frac{v-1}{4} & \text{if } 1 \leq v \leq 5 \\ 0 & \text{otherwise} \end{cases}$$



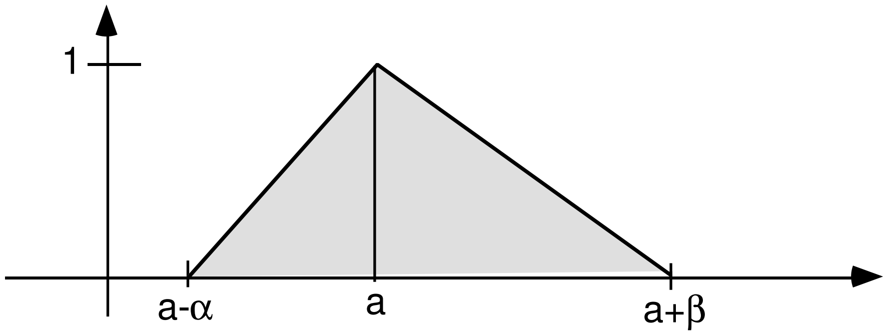
Membership function for "small volume".

Definition 2 *A fuzzy set A is called triangular fuzzy number with peak (or center) a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form*

$$A(t) = \begin{cases} 1 - (a-t)/\alpha & \text{if } a - \alpha \leq t \leq a \\ 1 - (t-a)/\beta & \text{if } a \leq t \leq a + \beta \\ 0 & \text{otherwise} \end{cases}$$

and we use the notation

$$A = (a, \alpha, \beta).$$



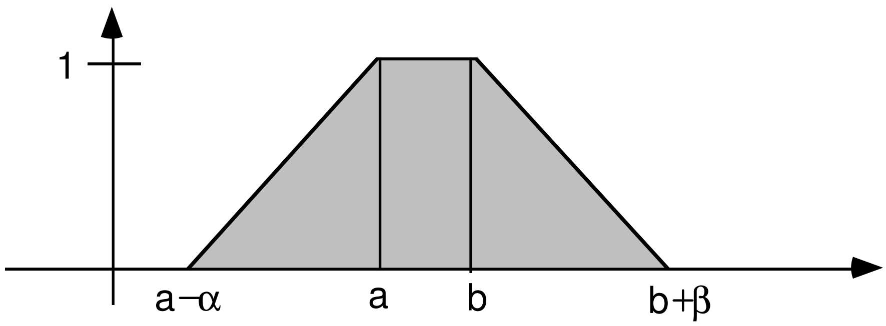
Triangular fuzzy number.

A triangular fuzzy number with center a may be seen as a fuzzy quantity

$$\text{"}x \text{ is approximately equal to } a\text{"}.$$

Definition 3 *A fuzzy set A is called trapezoidal fuzzy number with tolerance interval $[a, b]$, left width α and right width β if its membership function has the following form*

$$A(t) = \begin{cases} 1 - (a-t)/\alpha & \textit{if } a - \alpha \le t \le a \\ 1 & \textit{if } a \le t \le b \\ 1 - (t-b)/\beta & \textit{if } a \le t \le b + \beta \\ 0 & \textit{otherwise} \end{cases}$$



Trapezoidal fuzzy number.

A trapezoidal fuzzy number may be seen as a fuzzy quantity

$$\text{"}x \text{ is approximately in the interval } [a, b]\text{"}.$$

Definition 4 *Let A and B are fuzzy subsets of a classical set X. We say that A is a subset of B if $A(t) \leq B(t)$, $\forall t \in X$.*

Operations on fuzzy sets

In this section we extend the classical set theoretic operations from ordinary set theory to fuzzy sets.

We note that all those operations which are extensions of crisp concepts reduce to their usual meaning when the fuzzy subsets have membership degrees that are drawn from $\{0, 1\}$.

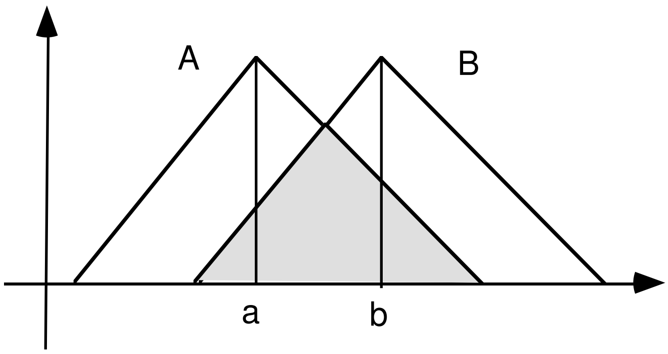
For this reason, when extending operations to fuzzy sets we use the same symbol as in set theory.

Let A and B are fuzzy subsets of a nonempty (crisp) set X.

Definition 5 *The intersection of A and B is defined as*

$$(\mathrm{A} \cap \mathrm{B})(\mathrm{t}) = \min\{\mathrm{A}(\mathrm{t}), \mathrm{B}(\mathrm{t})\} = \mathrm{A}(\mathrm{t}) \wedge \mathrm{B}(\mathrm{t}),$$

for all $t \in X$.



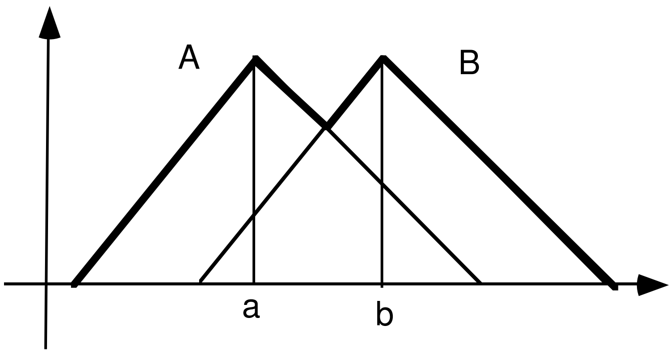
Intersection of two triangular fuzzy numbers.

$A \cap B$ can be interpreted as

"x is close to a **and** x is close to b"

Definition 6 *(union) The union of A and B is defined as*

$$(A \cup B)(t) = \max\{A(t), B(t)\} = A(t) \vee B(t), \ \forall t \in X$$



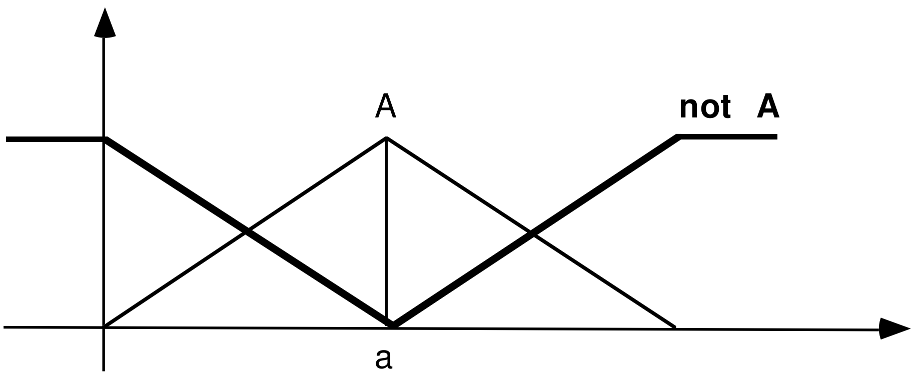
Union of two triangular fuzzy numbers.

$A \cup B$ can be interpreted as

"x is close to a **or** x is close to b"

Definition 7 *The complement of a fuzzy set A is defined as*

$$(\neg A)(t) = 1 - A(t)$$



Fuzzy sets A and $\neg A$.

A can be interpreted as

"x is close to a"

and $\neg A$ means

"x is not close to a"

or, in other words

"x is far from a"

Basics of fuzzy decision making

1 Bellman and Zadeh's principle

- R.A.Bellman and L.A.Zadeh, Decision-making in a fuzzy environment, *Management Sciences*, Ser. B **17**(1970) 141-164.

A classical MADM problem can be expressed in a matrix format. A decision matrix is an $m \times n$ matrix whose element x_{ij} indicates the performance rating of the i-th alternative, A_i, with respect to the j-th attribute, X_j:

$$\begin{array}{ccccc} & X_1 & X_2 & \dots & X_n \\ A_1 & x_{11} & x_{12} & \dots & x_{1n} \\ A_2 & x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ A_n & x_{n1} & x_{n2} & \dots & x_{nn} \end{array}$$

The classical *maximin* method is used to select an alternative A^* such that

$$A^* = \{A_k \mid \min_{1 \le j \le m} x_{kj} = \max_{1 \le i \le n} \min_{1 \le j \le m} x_{ij}\}$$

Example 3

	Mathematics	*English*	*History*
John	5	8	10
Mary	6	6	7
Kate	10	7	5

Mary is selected if we use the maximin method, because her minimal performance is the maximal.

The overall performance of an alternative is determined by its weakest performance.

A chain is only as strong as its weakest link.

The classical *maximax* method is used to select an alternative A^* such that

$$A^* = \{A_k \mid \max_{1\leq j\leq m} x_{kj} = \max_{1\leq i\leq n} \max_{1\leq j\leq m} x_{ij}\}$$

Example 4

	Mathematics	*English*	*History*
John	5	8	10
Mary	6	6	7
Kate	10	7	5

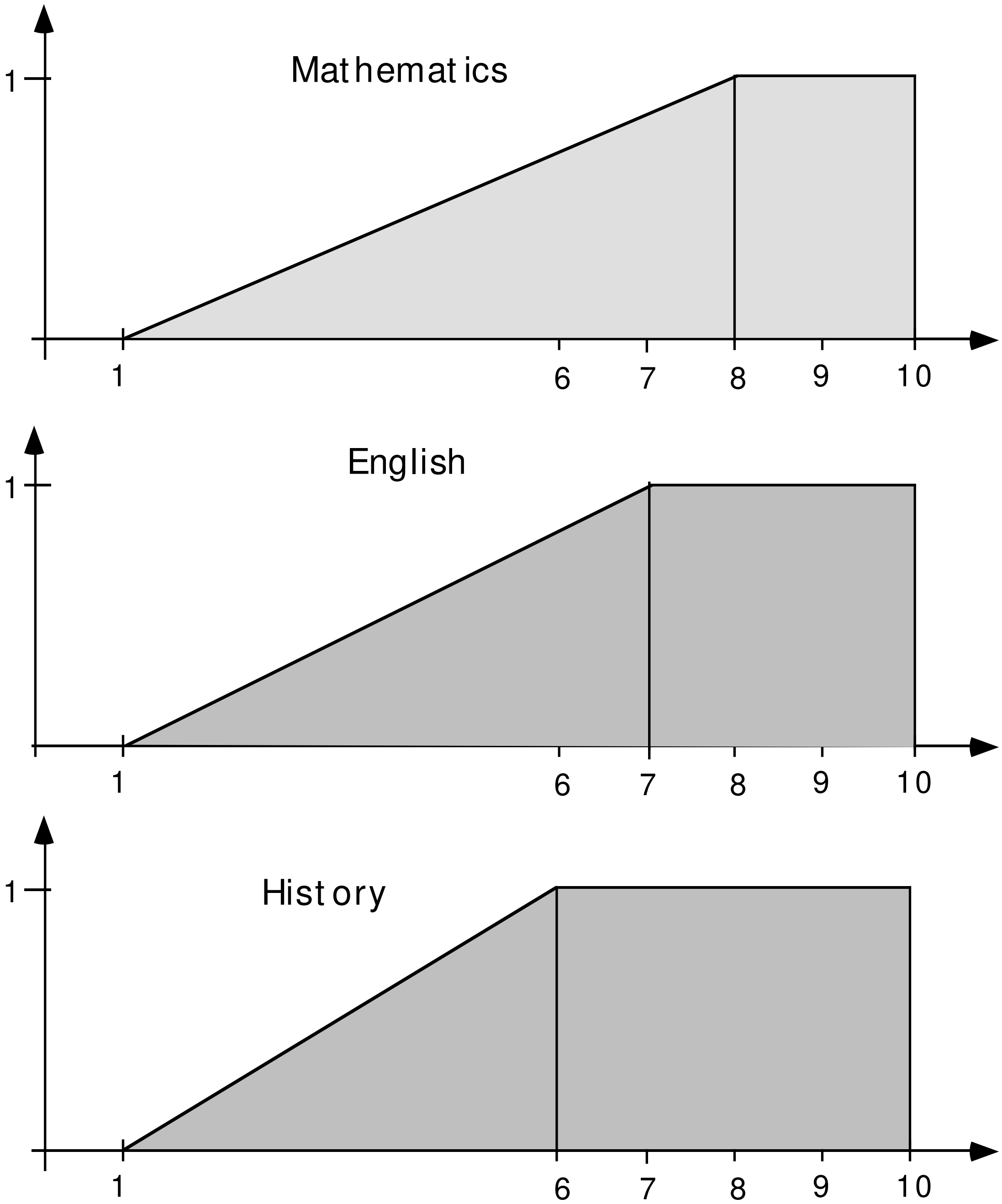
John or Kate are selected if we use the maximax method, because their maximal performances provide the global maximum.

In a decision process the idea of *trade-offs* corresponds to viewing the global evaluation of an action as lying between the *worst* and the *best* local ratings.
This occurs in the presence of conflicting goals, when a compensation between the corresponding compabilities is allowed.
Averaging operators realize trade-offs between objectives, by allowing a positive compensation between ratings.

A typical averaging operator is the *arithmetic mean.*

In fuzzy case the values of the decision matrix are given as degrees of "how an alternative satisfies a certain attribute".

For each attribute X_j we are given a fuzzy set μ_j measuring the degrees of satisfaction to the j-th attribute.



	Mathematics	*English*	*History*
John	5/7	1	1
Mary	6/7	5/6	1
Kate	1	1	4/5

Bellman and Zadeh's principle to fuzzy decision making chooses the "best compromise" alternative using the maximin method:

$$score(A_1) = \min\{5/7, 1, 1\} = 5/7$$

$$score(A_2) = \min\{6/7, 5/6, 1\} = 5/6$$

$$score(A_3) = \min\{1, 1, 4/5\} = 4/5$$

$$score(A_2) = \max\{score(A_1), score(A_2), score(A_3)\}$$

Mary is choosen as the best student.

If we have a continuous problem, which means that we have infinite number of alternatives then suppose that $C_1, C_2, \ldots, C_n$ are the membership functions coresponding to attributes $X_1, X_2, \ldots, X_n$, respectively.

Let x be an object such that for any criterion C_j, $C_j(x) \in [0, 1]$ indicates the degree to which this criterion is satisfied by x.

If we want to find out the degree to which x satisfies *"all* the criteria" denoting this by $D(x)$, we get following

$$D(x) = \min\{C_1(x), \ldots, C_n(x)\}$$

In this case we are essentially requiring x to satisfy

$$C_1 \text{ and } C_2 \text{ and } \ldots \text{ and } C_n$$

The best alternative, denoted by x^*, is determined from the relationship

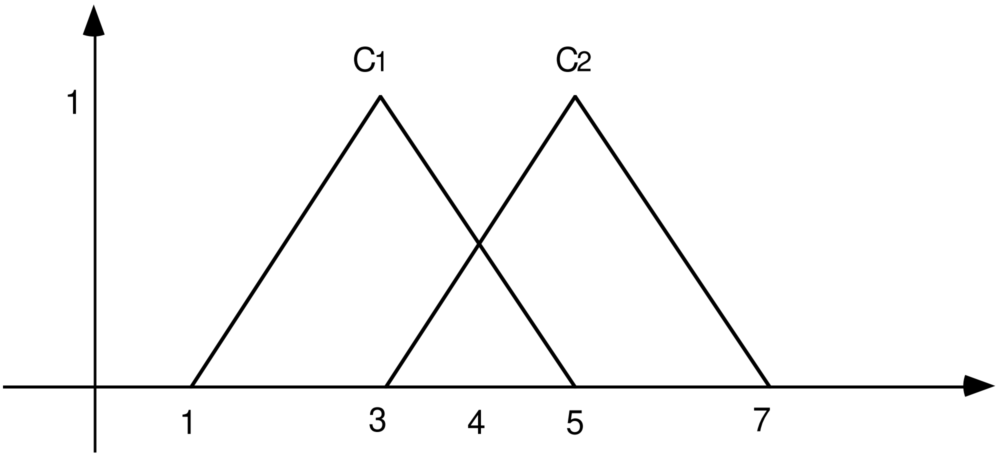
$$D(x^*) = \max_{x \in X} D(x)$$

This method is called *Bellman and Zadeh's principle to fuzzy decision making.*

If we desire to find out the degree to which x satisfies *"at least one* of the criteria", denoting this $E(x)$, we get

$$E(x) = \max\{C_1(x), \ldots, C_n(x)\}$$

In this case we are requiring x to satisfy C_1 or C_2 or $\ldots$ or C_n.



The first attribute called "x should be close to 3" is represented by the fuzzy set C_1 defined by

$$C_1(x) = \begin{cases} 1 - |3-x|/2 & \text{if } |3-x| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

The second attribute called "x should be close to 5" is represented by the fuzzy set C_2 defined by

$$C_2(x) = \begin{cases} 1 - |5-x|/2 & \text{if } |5-x| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

The fuzzy decision D is defined by the intersection of C_1 and C_2 and the membership function of D is

$$D(x) = \begin{cases} 1/2(1 - |4-x|) & \text{if } |4-x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The optimal solution is $x^* = 4$, because

$$D(4) = 1/2 = \max_x D(x).$$

In many applications rather than desiring that a solution satisfies one of these extreme situations, "all" or "at least one", we may require that x satisfies *most* or *at least half* of the criteria.