# Signal Processing First

**Lecture 17**
**IIR Filters: H(z) and Frequency Response**





# READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, Sects. 8-4 8-5 & 8-6
- Other Reading:
  - Recitation: Chapter 8, all
    - POLE-ZERO PLOTS





# LECTURE OBJECTIVES

- SYSTEM FUNCTION: H(z)
- H(z) has **POLES** and ZEROS
- FREQUENCY RESPONSE of IIR
  - Get H(z) first

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

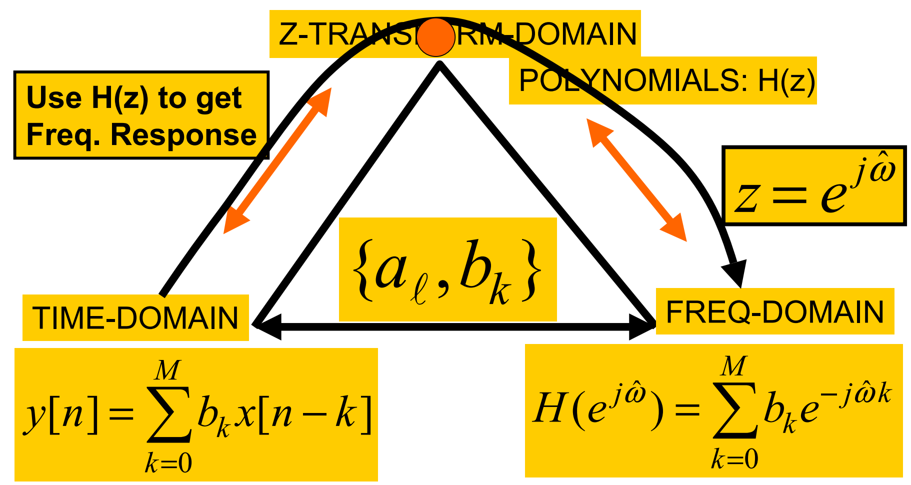
- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$





# THREE DOMAINS



# H(z) = z-Transform{ h[n] }

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$\boxed{h[n] = b_0 (a_1)^n u[n]}$$

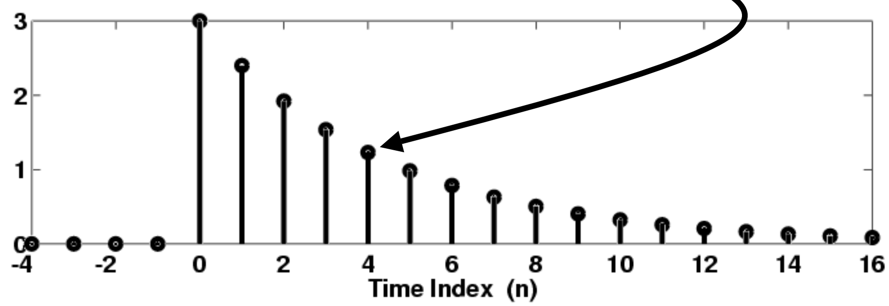
$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$





# PLOT IMPULSE RESPONSE

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$








# First-Order Transform Pair

$$h[n] = b a^n u[n] \quad \leftrightarrow \quad H(z) = \frac{b}{1 - a z^{-1}}$$

- GEOMETRIC SEQUENCE:

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$





# DELAY PROPERTY of X(z)

- DELAY in TIME<-->Multiply X(z) by $z^{-1}$

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1} X(z)$$

$$\text{Proof:} \quad \sum_{n=-\infty}^{\infty} x[n-1] z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-(\ell+1)}$$

$$= z^{-1} \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-\ell} = z^{-1} X(z)$$

# Z-Transform of IIR Filter

- DERIVE the SYSTEM FUNCTION H(z)
  - Use **DELAY** PROPERTY

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]}$$

$$\boxed{Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)}$$

**EASIER with DELAY PROPERTY**

*Time delay of $n_0$ samples multiplies the z-transform by $z^{-n_0}$*

$$x[n - n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$





# SYSTEM FUNCTION of IIR

- NOTE the FILTER COEFFICIENTS

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \boxed{\frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}} = \frac{B(z)}{A(z)}$$





# SYSTEM FUNCTION

- DIFFERENCE EQUATION:

$$\boxed{y[n] = 0.8 y[n-1] + 3x[n] - 2x[n-1]}$$

- **READ** the FILTER COEFFS:

**H(z)**

$$Y(z) = \left( \frac{3 - 2z^{-1}}{1 - 0.8z^{-1}} \right) X(z)$$





# CONVOLUTION PROPERTY

- **MULTIPLICATION** of z-TRANSFORMS

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) = H(z)X(z)$$

- **CONVOLUTION** in TIME-DOMAIN

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

**IMPULSE RESPONSE**

## POLES & ZEROS

- ROOTS of Numerator & Denominator

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow H(z) = \frac{b_0 z + b_1}{z - a_1}$$

$$b_0 z + b_1 = 0 \quad \Rightarrow z = -\frac{b_1}{b_0}$$

**ZERO: H(z)=0**

$$z - a_1 = 0 \quad \Rightarrow z = a_1$$

**POLE: H(z) → inf**



## EXAMPLE: Poles & Zeros

- VALUE of H(z) at POLES is **INFINITE**

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

$$H(z) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

**ZERO at z= -1**

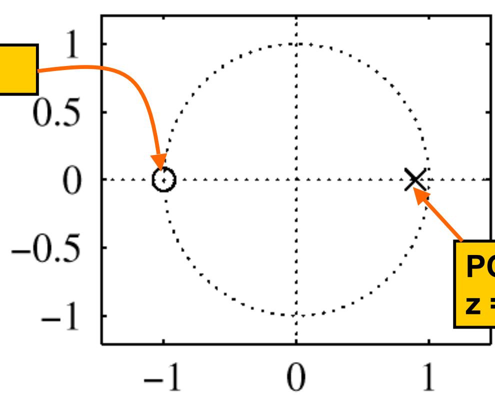
$$H(z) = \frac{2 + 2(\frac{4}{5})^{-1}}{1 - 0.8(\frac{4}{5})^{-1}} = \frac{\frac{9}{2}}{0} \rightarrow \infty$$

**POLE at z=0.8**



## POLE-ZERO PLOT

**ZERO at z = -1**




**POLE at z = 0.8**

$$\frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$



## FREQUENCY RESPONSE

- SYSTEM FUNCTION: H(z)
- H(z) has DENOMINATOR
- FREQUENCY RESPONSE of IIR
  - We have H(z)

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z = e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

# FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$





# FREQ. RESPONSE FORMULA

$$H(z) = \frac{1}{1-0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{1}{1-0.8e^{-j\hat{\omega}}}$$

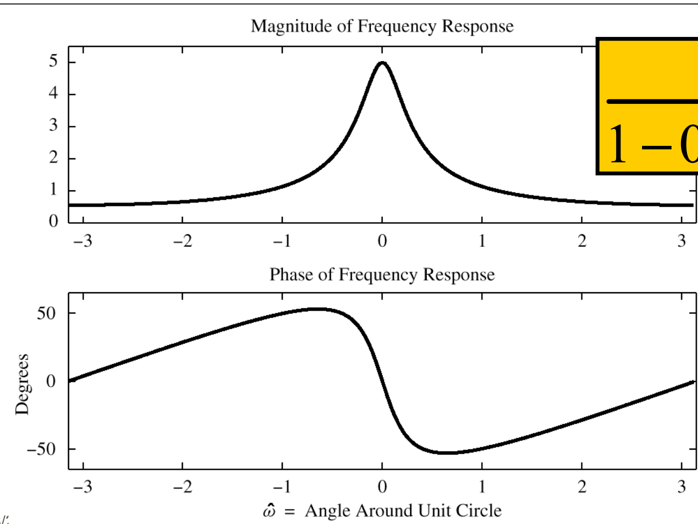
$$\left|H(e^{j\hat{\omega}})\right|^2 = \left|\frac{1}{1-0.8e^{-j\hat{\omega}}}\right|^2 = \frac{1}{1-0.8e^{-j\hat{\omega}}} \cdot \frac{1}{1-0.8e^{j\hat{\omega}}}$$

$$\frac{1}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}} = \frac{1}{1.64-1.6\cos\hat{\omega}}$$

$$@\hat{\omega}=0, \quad \left|H(e^{j\hat{\omega}})\right|^2 = \frac{1}{0.04} = 25 \qquad @\hat{\omega}=\pi?$$



# FREQ. RESPONSE from H(z)

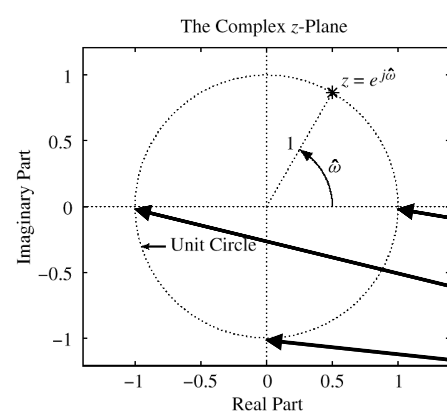



$$\frac{1}{1-0.8e^{-j\hat{\omega}}}$$




# UNIT CIRCLE

- MAPPING BETWEEN $z$ and $\hat{\omega}$



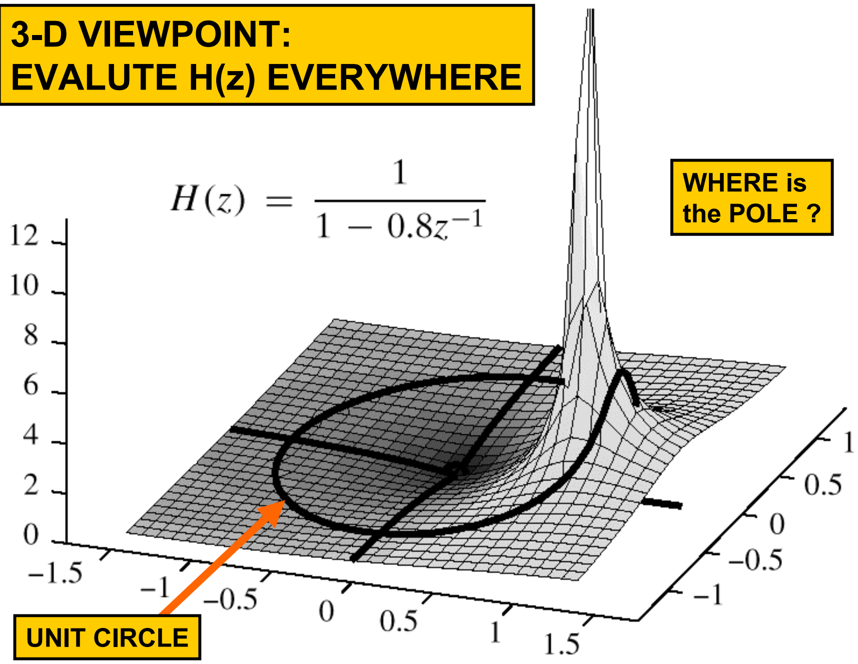

$$z = e^{j\hat{\omega}}$$

$$z = 1 \leftrightarrow \hat{\omega} = 0$$

$$z = -1 \leftrightarrow \hat{\omega} = \pm\pi$$

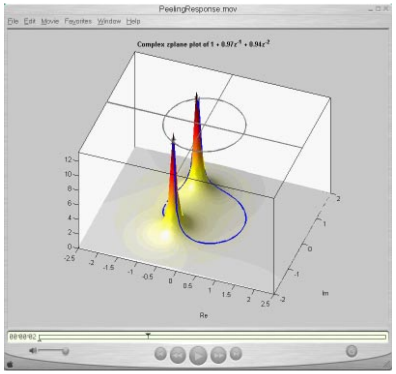
$$z = \pm j \leftrightarrow \hat{\omega} = \pm\tfrac{1}{2}\pi$$

**3-D VIEWPOINT:**
**EVALUTE H(z) EVERYWHERE**




# MOVIE for H(z) in 3-D

- POLES to H(z) to Frequency Reponse
  - TWO POLES SHOWN







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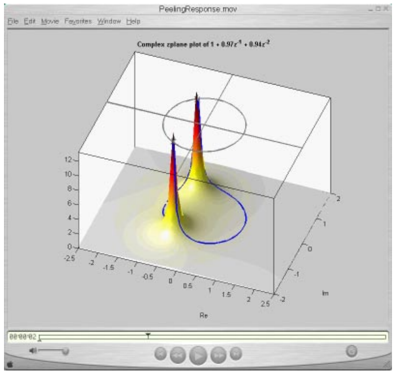
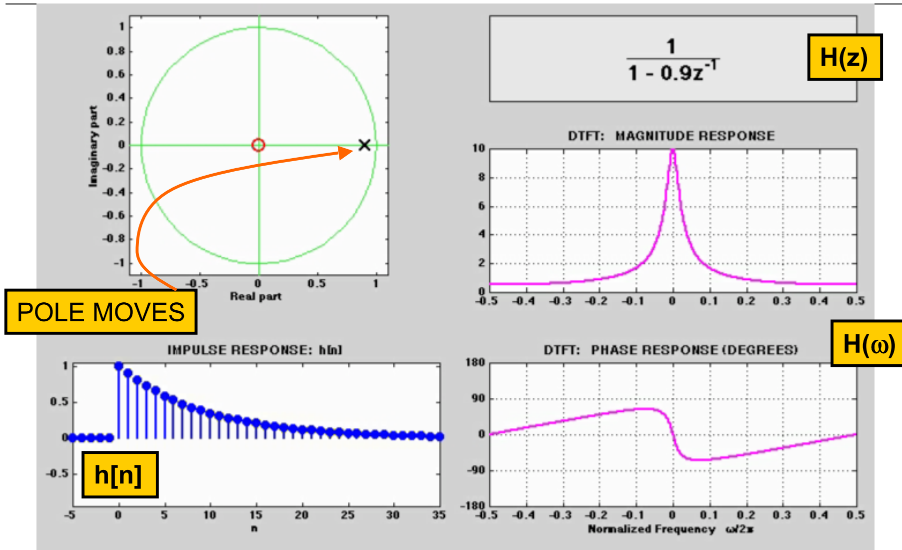


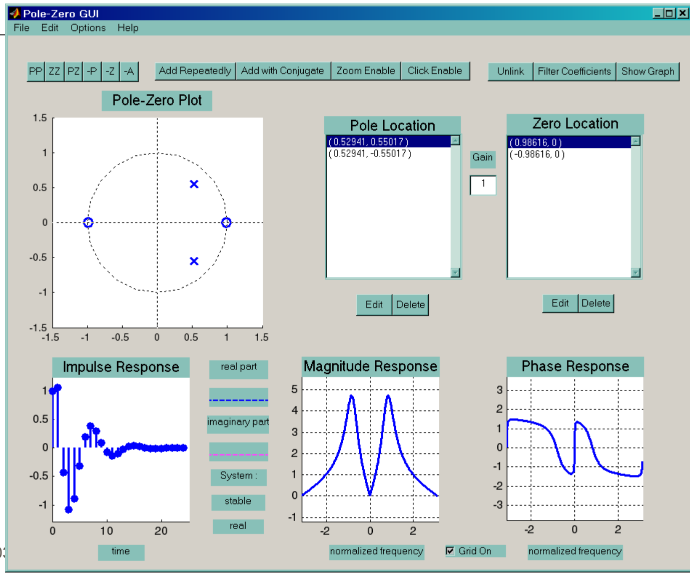




# 3 DOMAINS MOVIE: IIR



# PeZ Demo: Pole-Zero Placing






# SINUSOIDAL RESPONSE

- x[n] = SINUSOID => y[n] is SINUSOID
- Get MAGNITUDE & PHASE from H(z)

$$\text{if} \quad x[n] = e^{j\hat{\omega}n}$$

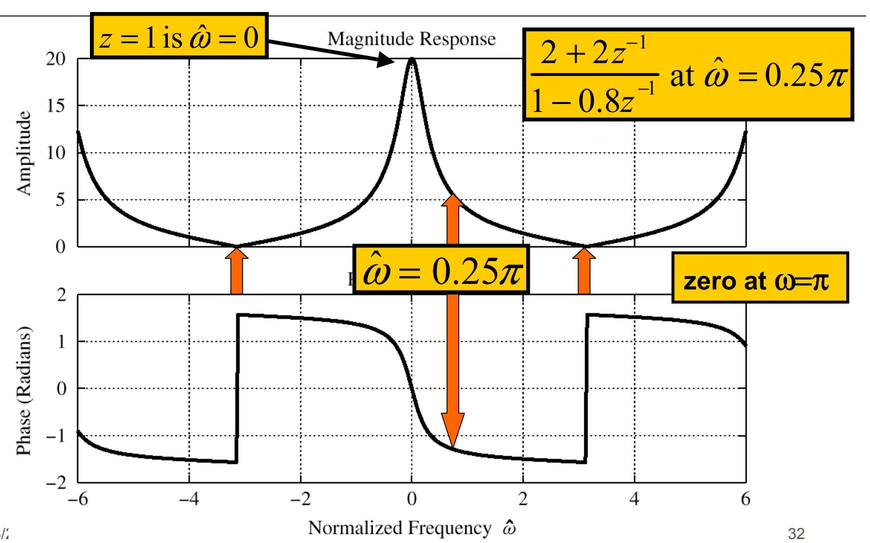
$$\text{then} \quad y[n] = H(e^{j\hat{\omega}})\, e^{j\hat{\omega}n}$$

$$\text{where} \quad H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$





# Evaluate FREQ. RESPONSE






# POP QUIZ: Eval Freq. Resp.

- Given: $H(z) = \dfrac{2+2z^{-1}}{1-0.8z^{-1}}$
- Find output, y[n], when $x[n] = \cos(0.25\pi n)$
  - Evaluate at $z = e^{j0.25\pi}$

$$H(z) = \frac{2+2\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right)}{1-0.8e^{-j0.25\pi}} = 5.182e^{-j1.309}$$

$$y[n] = 5.182\cos(0.25\pi n - 0.417\pi)$$