Signal Processing First

Lecture 18

3-Domains for IIR

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, all
- Other Reading:
  - Recitation: Ch. 8, all
    - POLES & ZEROS
  - Next Lecture: Chapter 9

LECTURE OBJECTIVES

- SECOND-ORDER IIR FILTERS
  - TWO FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

- H(z) can have COMPLEX POLES & ZEROS
- THREE-DOMAIN APPROACH
  - BPFs have POLES NEAR THE UNIT CIRCLE

THREE DOMAINS

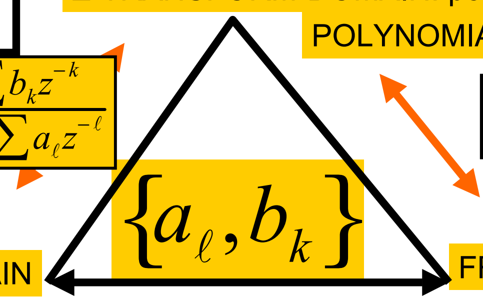
Use H(z) to get Freq. Response

Z-TRANSFORM-DOMAIN: poles & zeros

POLYNOMIALS: H(z)

$$H(z) = \frac{\sum b_k z^{-k}}{1 - \sum a_\ell z^{-\ell}}$$

$$\boxed{z = e^{j\hat{\omega}}}$$



$$\{a_\ell, b_k\}$$

TIME-DOMAIN

FREQ-DOMAIN

$$y[n] = a_1 y[n-1] + \sum_{k=0}^{M} b_k x[n-k]$$

Z-TRANSFORM TABLES

SHORT TABLE OF z-TRANSFORMS

	$x[n]$		$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\Longleftrightarrow$	$aX_1(z) + bX_2(z)$
2.	$x[n - n_0]$	$\Longleftrightarrow$	$z^{-n_0}X(z)$
3.	$y[n] = x[n] * h[n]$	$\Longleftrightarrow$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\Longleftrightarrow$	1
5.	$\delta[n - n_0]$	$\Longleftrightarrow$	z^{-n_0}
6.	$a^n u[n]$	$\Longleftrightarrow$	$\dfrac{1}{1 - az^{-1}}$

SECOND-ORDER FILTERS

- Two FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$\boxed{H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}}$$

MORE POLES

- Denominator is QUADRATIC
  - 2 Poles: REAL
  - or COMPLEX CONJUGATES

$$\boxed{\frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}}$$

$$\boxed{H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}}$$

PROPERTY OF REAL POLYNOMIALS

A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.

TWO COMPLEX POLES

- Find Impulse Response ?
  - Can OSCILLATE vs. n
  - "RESONANCE"

$$\boxed{(p_k)^n = (re^{j\theta})^n = r^n e^{jn\theta}}$$

- Find **FREQUENCY RESPONSE**
  - Depends on Pole Location
  - Close to the Unit Circle?
    - Make **BANDPASS FILTER**

$$\boxed{\begin{array}{l} \text{pole} = re^{j\theta} \\ r \to 1\,? \end{array}}$$

2nd ORDER EXAMPLE

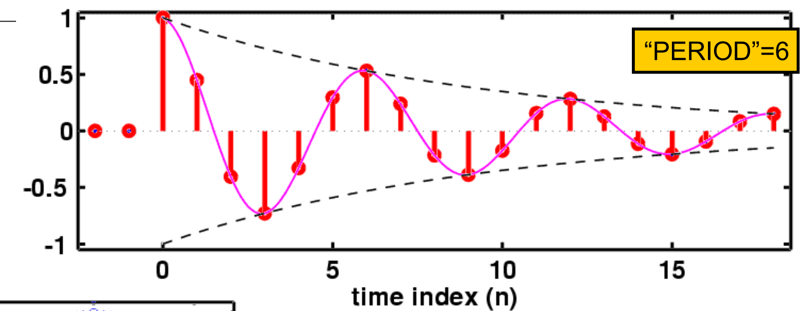
$$h[n]=(0.9)^n\cos(\tfrac{\pi}{3}n)u[n]=(0.9)^n\,\tfrac{1}{2}(e^{j\pi n/3}+e^{-j\pi n/3})u[n]$$

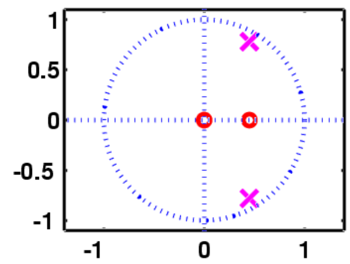
$$H(z)=\frac{0.5}{1-0.9e^{j\pi/3}z^{-1}}+\frac{0.5}{1-0.9e^{-j\pi/3}z^{-1}}$$

$$H(z)=\frac{1-0.9\cos(\frac{\pi}{3})z^{-1}}{(1-0.9e^{j\pi/3}z^{-1})(1-0.9e^{-j\pi/3}z^{-1})}$$

$$H(z)=\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

h[n]: Decays & Oscillates





$$h[n]=(0.9)^n\cos(\tfrac{\pi}{3}n)u[n]$$

$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

2nd ORDER Z-transform PAIR

GENERAL ENTRY for z-Transform TABLE

$$h[n]=r^n\cos(\theta n)u[n]$$

$$H(z)=\frac{1-r\cos\theta z^{-1}}{1-2r\cos\theta z^{-1}+r^2z^{-2}}$$

$$h[n]=Ar^n\cos(\theta n+\varphi)u[n]$$

$$H(z)=A\frac{\cos\varphi-r\cos(\theta-\varphi)z^{-1}}{1-2r\cos\theta z^{-1}+r^2z^{-2}}$$

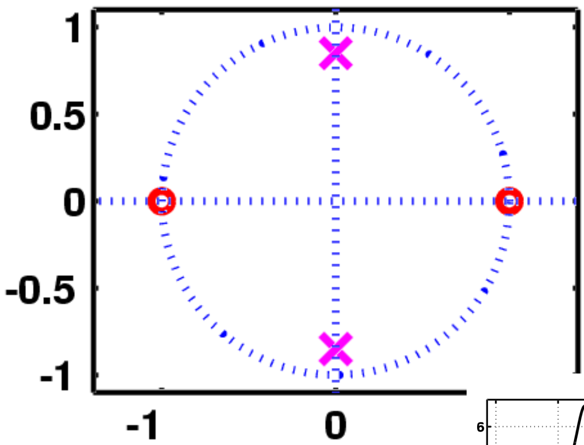
2nd ORDER EX: n-Domain

$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

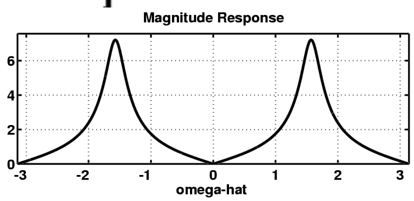
$$y[n]=0.9y[n-1]-0.81y[n-2]+x[n]-0.45x[n-1]$$

```
aa = [ 1, -0.9, 0.81 ];
bb = [ 1, -0.45 ];
nn = -2:19;
hh = filter( bb, aa, (nn==0) );
HH = freqz( bb, aa, [-pi,pi/100:pi] );
```

Complex POLE-ZERO PLOT

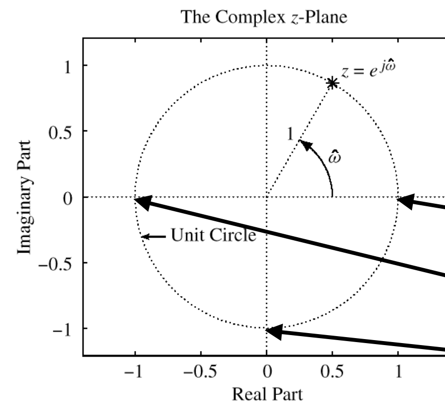


$$\frac{1-z^{-2}}{1+0.7225z^{-2}}$$



UNIT CIRCLE

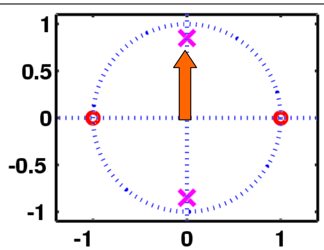
- MAPPING BETWEEN z and $\hat{\omega}$



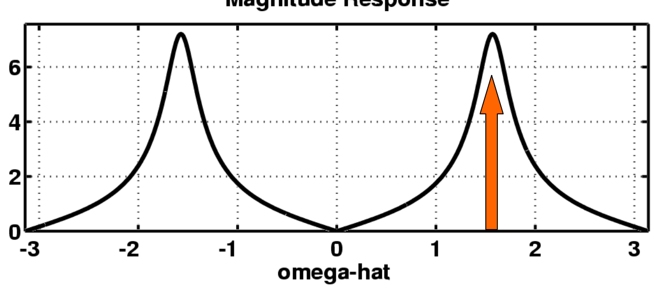
$$z = e^{j\hat{\omega}}$$

$$z = 1 \leftrightarrow \hat{\omega} = 0$$
$$z = -1 \leftrightarrow \hat{\omega} = \pm\pi$$
$$z = \pm j \leftrightarrow \hat{\omega} = \pm\tfrac{1}{2}\pi$$

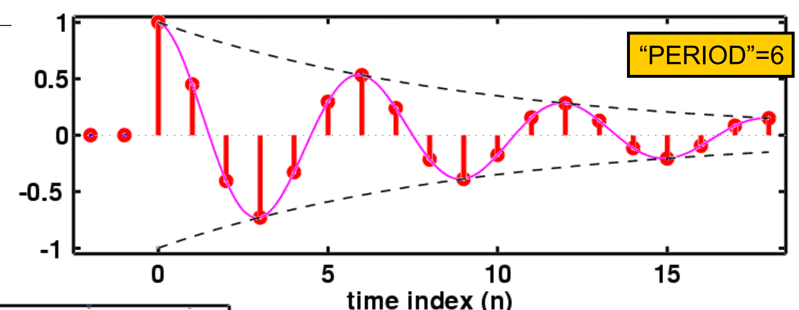
FREQUENCY RESPONSE from POLE-ZERO PLOT

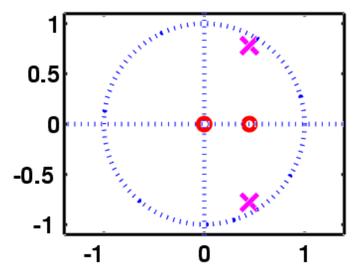


$$H(e^{j\hat{\omega}}) = \frac{1-e^{-j2\hat{\omega}}}{1+0.7225e^{-j2\hat{\omega}}}$$



h[n]: Decays & Oscillates

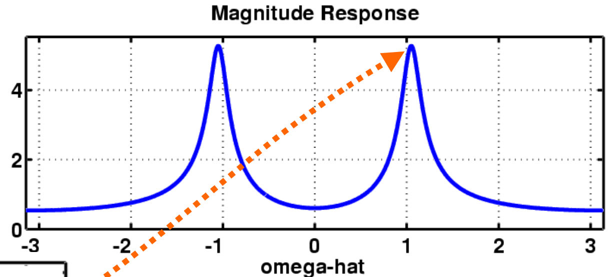


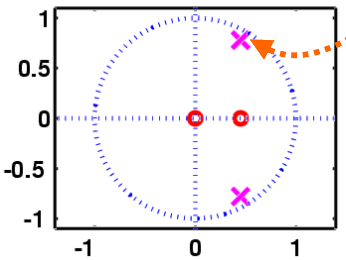


$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3}n)u[n]$$

$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

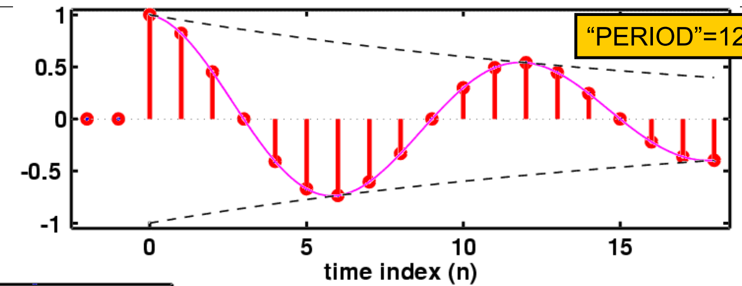
Complex POLE-ZERO PLOT

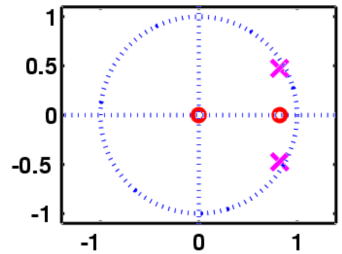




$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

h[n]: Decays & Oscillates

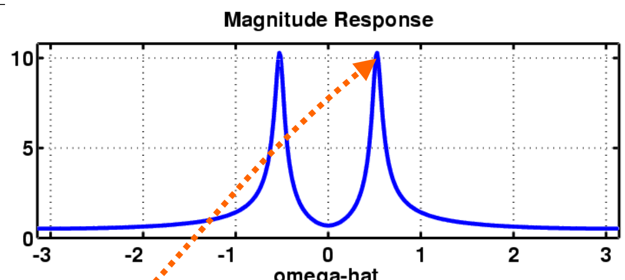


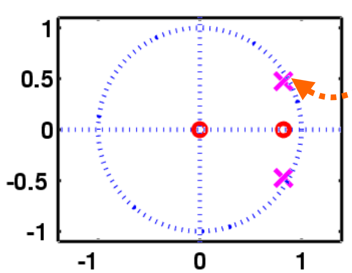


$$h[n]=(0.95)^n\cos(\tfrac{\pi}{6}n)u[n]$$

$$\frac{1-0.8227z^{-1}}{1-1.6454z^{-1}+0.9025z^{-2}}$$

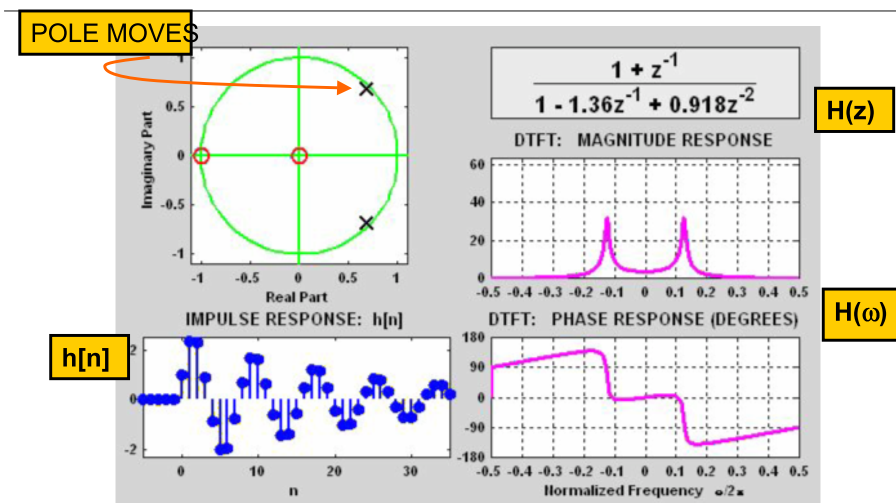
Complex POLE-ZERO PLOT





$$\frac{1-0.8227z^{-1}}{1-1.6454z^{-1}+0.9025z^{-2}}$$

3 DOMAINS MOVIE: IIR



THREE INPUTS

- Given: $\longrightarrow \boxed{H(z) = \dfrac{5}{1+0.8z^{-1}}} \longrightarrow$
- Find the output, y[n]
  - When

$$x[n] = \cos(0.2\pi n)$$

$$x[n] = u[n]$$

$$x[n] = \cos(0.2\pi n)u[n]$$

SINUSOID ANSWER

- Given: $H(z) = \dfrac{5}{1+0.8z^{-1}}$
- The input: $x[n] = \cos(0.2\pi n)$
- Then y[n] $y[n] = M\cos(0.2\pi n + \psi)$

$$H(e^{j0.2\pi}) = \frac{5}{1+0.8e^{-j0.2\pi}} = 2.919e^{j0.089\pi}$$

Step Response

$$Y(z) = H(z)X(z) = \left(\frac{5}{1+.8z^{-1}}\right)\left(\frac{1}{1-z^{-1}}\right)$$

Partial Fraction Expansion

$$Y(z) = \frac{A}{1+.8z^{-1}} + \frac{B}{1-z^{-1}} = \frac{(A+B)+(.8B-A)z^{-1}}{(1+.8z^{-1})(1-z^{-1})}$$

$$\Rightarrow \quad (A+B) = 5 \quad \text{and} \quad (.8B-A) = 0$$

$$Y(z) = \frac{A}{1+.8z^{-1}} + \frac{B}{1-z^{-1}}$$

Step Response

$$Y(z) = \frac{\frac{20}{9}}{1+.8z^{-1}} + \frac{\frac{25}{9}}{1-z^{-1}}$$

$$y[n] = \frac{20}{9}(-.8)^n u[n] + \frac{25}{9}u[n]$$

$$y[n] \rightarrow \frac{25}{9} \quad \text{as} \quad n \rightarrow \infty$$

Stability

- Nec. & suff. condition: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

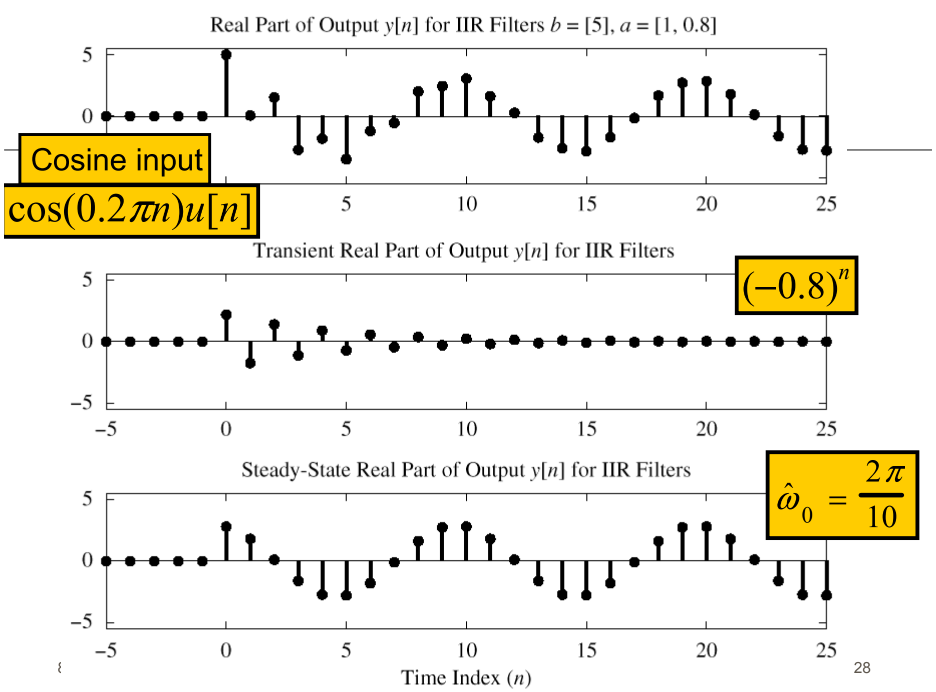
$$h[n] = b(a)^n u[n] \Leftrightarrow H(z) = \frac{b}{1 - az^{-1}}$$

$$\sum_{n=0}^{\infty} |b||a|^n < \infty \text{ if } |a| < 1 \Rightarrow$$

Pole must be Inside unit circle

SINUSOID starting at n=0

- We'll look at an example in MATLAB
  - $\cos(0.2\pi n)$
  - Pole at –0.8, so a^n is $(-0.8)^n$
- There are two components:
  - TRANSIENT
    - Start-up region just after n=0; **$(-0.8)^n$**
  - STEADY-STATE
    - Eventually, y[n] looks sinusoidal.
    - **Magnitude & Phase from Frequency Response**



STABILITY

- When Does the TRANSIENT DIE OUT ?

STEADY-STATE RESPONSE AND STABILITY

A stable system is one that does not "blow up." This intuitive statement can be formalized by saying that the output of a stable system can always be bounded ($|y[n]| < M_y$) whenever the input is bounded ($|x[n]| < M_x$).[3]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

$$h[n] = b_0 a_1^n u[n]$$

$$\text{need } |a_1| < 1$$

STABILITY CONDITION

- ALL POLES INSIDE the UNIT CIRCLE
- UNSTABLE EXAMPLE:

POLE @ z=1.1

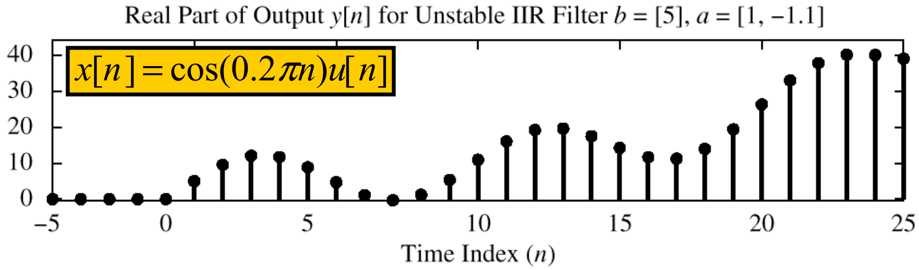


Figure 8.15 Illustration of an unstable IIR system. Pole is at $z = 1.1$.

BONUS QUESTION

- Given: $H(z) = \dfrac{5}{1 + 0.8z^{-1}}$
- The input is $x[n] = 4\cos(\pi n - 0.5\pi)$
- Then find y[n] $y[n] = ?$