PROBLEM 8.1:

$$y[n] = \sqrt{2}\, y[n-1] - y[n-2] + x[n]$$

$x[n] = \delta[n]$

"At rest" condition $\Rightarrow$ $y[n]=0$ for $n<0$.

$$y[0] = \sqrt{2}\, y[-1] - y[-2] + x[0] = (\sqrt{2})0 - 0 + 1 = 1$$
$$y[1] = \sqrt{2}\, y[0] - y[-1] + x[1] = (\sqrt{2})1 - 0 + 0 = \sqrt{2}$$
$$y[2] = \sqrt{2}\, y[1] - y[0] + x[2] = (\sqrt{2})\sqrt{2} - 1 + 0 = 1$$
$$y[3] = (\sqrt{2})1 - \sqrt{2} + 0 = 0$$
$$y[4] = (\sqrt{2})0 - 1 + 0 = -1$$

The general formula is

$$y[n] = A_1 (r_1)^n + A_2 (r_2)^n \quad \text{for } n \geq 0$$

where r_1 & r_2 are the poles.

$$H(z) = \frac{1}{1 - \sqrt{2}\, z^{-1} + z^{-2}}$$

Poles are roots of denominator:

$$\frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2} = e^{\pm j\pi/4}$$

$$y[n] = A_1 (e^{j\pi/4})^n + A_2 e^{-j\frac{\pi}{4}n}$$

Now, we evaluate A_1 & A_2 from known values of $y[n]$. We use $n=2$ and $n=4$

$$y[2] = 1 = A_1 e^{j\pi/2} + A_2 e^{-j\pi/2} = jA_1 - jA_2$$
$$y[4] = -1 = A_1 e^{j\pi} + A_2 e^{-j\pi} = -A_1 - A_2$$

Solve the simultaneous equations:

$$1-j = -2jA_2 \quad \text{and} \quad 1+j = 2jA_1 \Rightarrow A_1 = \frac{1+j}{2j} = \frac{1}{2} - j\frac{1}{2}$$

$$A_2 = A_1^* \qquad A_1 = \frac{\sqrt{2}}{2} e^{-j\pi/4}$$

$$y[n] = \frac{\sqrt{2}}{2} e^{-j\pi/4} e^{j\frac{\pi}{4}n} + \frac{\sqrt{2}}{2} e^{+j\pi/4} e^{-j\pi n/4} \quad \text{for } n \geq 0$$

$$= \sqrt{2}\cos\left(\frac{\pi}{4}n - \frac{\pi}{4}\right) = \sqrt{2}\cos\left(\frac{\pi}{4}(n-1)\right)$$

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PROBLEM 8.6:

(a) $y[n] = -\frac{1}{2}y[n-1] + x[n]$

If re-arranged:

$$y[n] + \frac{1}{2}y[n-1] = x[n]$$

$\underline{a} = [1 \ \ +\frac{1}{2}];$
$b = 1$

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} = \frac{z}{z + 1/2}$$

ZERO @ $z = 0$
POLE @ $z = -\frac{1}{2}$

(b) Do this by making a table.

n	$x[n]$	$y[n]$	
<0	0	0	
0	1	1	$y[0] = -\frac{1}{2}y[-1] + x[0] = 1$
1	1	$1/2$	$y[1] = -\frac{1}{2}(1) + 1 = 1/2$
2	1	$3/4$	$y[2] = -\frac{1}{2}(1/2) + 1 = 3/4$
3	0	$-3/8$	$y[3] = -1/2\,(3/4) + 0$
4	0	$3/16$	$y[4] = -1/2\,(-3/8) + 0$ -
5	0	$-3/32$	
6	0	$3/64$	
7	0	$-3/128$	
8	0	$3/256$	

$$\therefore \quad y[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ 1/2 & n = 1 \\ 3/(-2)^n & n \geq 2 \end{cases}$$

RESPONSE BEHAVES LIKE

$3\left(\frac{-1}{2}\right)^n$ for $n \geq 2$

POLE

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PROBLEM 8.13:

Characterize each system ($S_1 \to S_7$)

S_1: $H_1(z) = \dfrac{\frac{1}{2} + \frac{1}{2}z^{-1}}{1 - 0.9z^{-1}}$ $\Rightarrow$ pole at $z = 0.9$
zero at $z = -1$

$H_1(e^{j\hat{\omega}})$ is a LPF with a null at $\hat{\omega} = \pi$.

S_2: $H_2(z) = \dfrac{9 + 10z^{-1}}{1 + 0.9z^{-1}}$ $\Rightarrow$ pole at $z = -0.9$
zero at $z = -10/9$

$H_2(e^{j\hat{\omega}})$ is an all-pass filter

S_3: $H_3(z) = \dfrac{\frac{1}{2}(1 - z^{-1})}{1 + 0.9z^{-1}}$ $\Rightarrow$ pole at $z = -0.9$
zero at $z = 1$

$H_3(e^{j\hat{\omega}})$ is a HPF with a null at $\hat{\omega} = 0$.

S_4: $H_4(z) = \frac{1}{4}(1 + 4z^{-1} + 6z^{-2} + 4z^{-3} + z^{-4})$

$= \frac{1}{4}(1 + z^{-1})^4$ $\Rightarrow$ 4 zeros at $z = -1$

$H_4(e^{j\hat{\omega}})$ is a LPF with null at $\hat{\omega} = \pi$.

DC value: $H_4(e^{j0}) = 4$.

S_5: $H_5(z) = 1 - z^{-1} + z^{-2} - z^{-3} + z^{-4} = \dfrac{1 + z^{-5}}{1 + z^{-1}}$

has 4 zeros around the unit circle.
No zero at $z = -1$; others at $e^{j(2\pi k/5 - \pi/5)}$

$H_5(e^{j\hat{\omega}})$ is a HPF with nulls at $\hat{\omega} = \pm\frac{\pi}{5}, \pm\frac{3\pi}{5}$

S_6: $H_6(z) = 1 + z^{-1} + z^{-2} + z^{-3} = \dfrac{1 - z^{-4}}{1 - z^{-1}}$

has 3 zeros around the unit circle at $z = \pm j, -1$

$H_6(e^{j\hat{\omega}})$ is a LPF with nulls at $\hat{\omega} = \pm\frac{\pi}{2}, \pi$

S_7: $H_7(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} = \dfrac{1 - z^{-6}}{1 - z^{-1}}$

has 5 zeros around the unit circle at $z = e^{j\pi k/3}$

$H_7(e^{j\hat{\omega}})$ is a LPF with nulls at $\hat{\omega} = \pm\frac{\pi}{3}, \pm 2\frac{\pi}{3}, \pi$

PZ#1: S_7 PZ#3: S_2 PZ#5: S_5

PZ#2: S_1 PZ#4: S_6 PZ#6: S_3

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PROBLEM 8.16:

PZ#1: zero at $z=1$ $\Rightarrow$ zero at $\hat{\omega}=0$

only (D) has a zero at DC

PZ#2: pole on real axis but far from $z=1$.

$\Rightarrow$ LPF with very wide passband. (B)

PZ#3: pole very close to $z=1$ $\Rightarrow$ narrow LPF

also, zero at $z=-1$ $\Rightarrow$ zero at $\hat{\omega}=\pi$ (A)

PZ#4: pole angles are approximately $\pm\pi/6$

$\Rightarrow$ peaks near $\hat{\omega}=\pm\pi/6$ (E)

PROBLEM 8.20:

Using $f_s = 1000$ samples/sec, we can determine $x[n]$.

$$x[n] = x(t)\Big|_{t=n/f_s} = 4 + \cos\left(500\pi \frac{n}{1000}\right) - 3\cos\left(2000\frac{\pi}{3}\cdot\frac{n}{1000}\right)$$

$$x[n] = 4 + \cos\left(\frac{\pi}{2}n\right) - 3\cos\left(\frac{2\pi}{3}n\right)$$

Use the frequency response at $\hat{\omega} = 0, \frac{\pi}{2}$ and $\frac{2\pi}{3}$ to determine $y[n]$:

$$y[n] = 4H(e^{j0}) + |H(e^{j\pi/2})|\cos\left(\frac{\pi}{2}n + \angle H(e^{j\pi/2})\right)$$
$$-3|H(e^{j2\pi/3})|\cos\left(\frac{2\pi}{3}n + \angle H(e^{j2\pi/3})\right)$$

Since $H(z)$ has zeros at $z=1$ and $z = e^{\pm j\pi/2}$, the frequency response is zero at $\hat{\omega}=0$ & $\hat{\omega}=\pi/2$

Thus we only need $H(e^{j2\pi/3})$:

$$H(e^{j2\pi/3}) = \frac{(1-e^{-j2\pi/3})(1-e^{j\pi/2}e^{-j2\pi/3})(1-e^{-j\pi/2}e^{-j2\pi/3})}{(1-0.9)(1-0.9e^{-j4\pi/3})}$$

$$= 10.522e^{j0.657\pi}$$

ANGLE = 118.26° or 2.064 rads

$$y[n] = -3(10.522)\cos\left(\frac{2\pi}{3}n + 0.657\pi\right)$$
$$= 31.566\cos\left(\frac{2\pi}{3}n - 0.343\pi\right)$$

INCORPORATE MINUS SIGN IN THE PHASE

Now convert back to continuous-time.

$$y(t) = y[n]\Big|_{n \leftarrow f_s t} = 31.566\cos\left(\frac{2\pi}{3}(1000)t - 0.343\pi\right)$$

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PROBLEM 9.2:

(a) An *exponentiation system* is defined by the input/output relation $y(t) = \exp\{x(t+2)\} = e^{x(t+2)}$

(i) *Linear:* The system is ***not*** linear because the sum of two inputs will give an output that is the product of the corresponding outputs:

$$\begin{aligned} x_1(t) &\rightarrow y_1(t) = e^{x_1(t+2)} \\ x_2(t) &\rightarrow y_2(t) = e^{x_2(t+2)} \\ x_1(t) + x_2(t) &\rightarrow e^{x_1(t+2)+x_2(t+2)} = e^{x_1(t+2)}e^{x_2(t+2)} = y_1(t)y_2(t) \end{aligned}$$

(ii) *Time-invariant:* The system is time-invariant because the system definition is a point-wise operator:

$$\begin{aligned} x_1(t) &\rightarrow y_1(t) = e^{x_1(t+2)} \\ x_1(t-t_1) &\rightarrow y_2(t) = e^{x_1(t+2-t_1)} = e^{x_1((t-t_1)+2)} = y_1(t-t_1) \end{aligned}$$

(iii) *Stable:* The system is stable because the system definition is a point-wise operator. If the input signal is bounded by M_x, i.e., $\max\{|x[n]|\} < M_x$, then the output signal is bounded by $M_y = e^{M_x}$.

(iv) *Causal:* The system is ***not*** causal because the system definition involves a time-shift of $(t+2)$ which is a shift by -2. Here is a counter-example:

$$x_1(t) = u(t) \rightarrow y_1(t) = e^{u(t+2)} = e^1 u(t+2)$$

In other words, the input "starts" at $t = 0$, while the output "starts earlier" at $t = -2$.

(b) A *phase modulator* is a system whose input and output satisfy a relation of the form $y(t) = \cos[\omega_c t + x(t)]$

(i) *Linear:* The system is ***not*** linear because the sum of two inputs will give an output that is the not the sum of the corresponding outputs. Let one of the input signals be the zero signal to get a counterexample:

$$\begin{aligned} x_1(t) &\rightarrow y_1(t) = \cos[\omega_c t + x_1(t)] \\ x_2(t) = 0 &\rightarrow y_2(t) = \cos[\omega_c t + x_2(t)] = \cos[\omega_c t] \\ x_1(t) + x_2(t) &\rightarrow \cos[\omega_c t + x_1(t) + x_2(t)] = \cos[\omega_c t + x_1(t)] = y_1(t) \neq y_1(t) + y_2(t) \end{aligned}$$

(ii) *Time-invariant:* The system is ***not*** time-invariant because the system definition contains a component that does not depend on $x(t)$. Here is a counterexample with a unit-step signal:

$$\begin{aligned} x_1(t) = \pi u(t) &\rightarrow y_1(t) = \cos[\omega_c t + \pi u(t)] = \cos[\omega_c t]u(-t) - \cos[\omega_c t]u(t) \\ x_1(t-1) = \pi u(t-1) &\rightarrow y_2(t) = \cos[\omega_c t + \pi u(t-1)] = \cos[\omega_c t]u(1-t) - \cos[\omega_c t]u(t-1) \\ \text{but, } y_1(t-1) &= \cos[\omega_c(t-1)]u(1-t) - \cos[\omega_c(t-1)]u(t-1) \end{aligned}$$

Thus, $y_2(t) \neq y_1(t-1)$ which means that $y_2(t)$ is not a shifted version of $y_1(t)$.

PROBLEM 9.2 (more):

(iii) *Stable:* The system is stable because the output will always be bounded by one, independent of the values of $x(t)$.

(iv) *Causal:* The system is causal because the output $y(t)$ depends only on the value of the input $x(t)$ **at the same time**. No values of $x(t)$ from the future (or the past) are used.

(c) An *amplitude modulator* is a system whose input and output satisfy a relation of the form $y(t) = [A + x(t)]\cos(\omega_c t)$

(i) *Linear:* The system is ***not*** linear because the sum of two inputs will give an output that is the not the sum of the corresponding outputs. Let one of the input signals be the zero signal to get a counterexample:

$$\begin{aligned} x_1(t) &\rightarrow y_1(t) = [A + x_1(t)]\cos(\omega_c t) \\ x_2(t) = 0 &\rightarrow y_2(t) = [A + x_2(t)]\cos(\omega_c t) = A\cos[\omega_c t] \\ x_1(t) + x_2(t) &\rightarrow [A + x_1(t) + x_2(t)]\cos(\omega_c t) = [A + x_1(t)]\cos(\omega_c t) = y_1(t) \neq y_1(t) + y_2(t) \end{aligned}$$

(ii) *Time-invariant:* The system is ***not*** time-invariant because the system definition contains a component that does not depend on $x(t)$. Here is a counterexample with a unit-step signal:

$$\begin{aligned} x_1(t) = -Au(t) &\rightarrow y_1(t) = [A - Au(t)]\cos(\omega_c t) = A\cos[\omega_c t]u(-t) \\ x_1(t-1) = Au(t-1) &\rightarrow y_2(t) = [A - Au(t-1)]\cos(\omega_c t) = A\cos[\omega_c t]u(1-t) \\ \text{but, } y_1(t-1) &= A\cos[\omega_c(t-1)]u(1-t) \end{aligned}$$

Thus, $y_2(t) \neq y_1(t-1)$ which means that $y_2(t)$ is not a shifted version of $y_1(t)$.

(iii) *Stable:* The system is stable because the output will always be bounded by $|A + \max\{|x[n]|\}|$.

(iv) *Causal:* The system is causal because the output $y(t)$ depends only on the value of the input $x(t)$ **at the same time**. No values of $x(t)$ from the future (or the past) are used.

PROBLEM 9.2 (more):

(d) A system that takes the even part of an input signal is defined by a relation of the form $y(t) = \mathcal{E}v\{x(t)\} = \dfrac{x(t) + x(-t)}{2}$

(i) *Linear:* The system is linear, so we have to prove both the scaling property and the superposition property:

$$\begin{aligned} x_1(t) &\rightarrow y_1(t) = \tfrac{1}{2}x_1(t) + \tfrac{1}{2}x_1(-t) \\ x_2(t) &\rightarrow y_2(t) = \tfrac{1}{2}x_2(t) + \tfrac{1}{2}x_2(-t) \\ x_1(t) + x_2(t) &\rightarrow \tfrac{1}{2}(x_1(t) + x_2(t)) + \tfrac{1}{2}(x_1(-t) + x_2(-t)) \\ &= \tfrac{1}{2}(x_1(t) + x_1(-t)) + \tfrac{1}{2}(x_2(t) + x_2(-t)) = y_1(t) + y_2(t) \\ \beta x_1(t) &\rightarrow \tfrac{1}{2}(\beta x_1(t)) + \tfrac{1}{2}(\beta x_1(-t)) = \beta\left(\tfrac{1}{2}x_1(t) + \tfrac{1}{2}x_1(-t)\right) = \beta y_1(t) \end{aligned}$$

(ii) *Time-invariant:* The system is ***not*** time-invariant because the system definition contains a flip. Here is a counterexample with a unit-impulse signal:

$$\begin{aligned} x_1(t) = \delta(t) &\rightarrow y_1(t) = \tfrac{1}{2}\{\delta(t) + \delta(-t)\} = \delta(t) \\ x_1(t-1) = \delta(t-1) &\rightarrow y_1(t) = \tfrac{1}{2}\{\delta(t-1) + \delta(-t-1))\} = \tfrac{1}{2} \\ delta(t-1) + \tfrac{1}{2}\delta(t+1)\} \\ \text{but, } y_1(t-1) &= \delta(t-1) \end{aligned}$$

Thus, $y_2(t) \neq y_1(t-1)$ which means that $y_2(t)$ is not a shifted version of $y_1(t)$.

(iii) *Stable:* The system is stable because the output will always be bounded by $\max\{|x[n]|\}|$.

$$\max\{|y[n]|\} = \max\{|\tfrac{1}{2}x(t) + \tfrac{1}{2}x(-t)|\} \leq \tfrac{1}{2}\max\{|x[n]|\}| + \tfrac{1}{2}\max\{|x[n]|\}|$$

(iv) *Causal:* The system is ***not*** causal because the flip component of the system definition creates a component in negative time. The signal $\delta(t-1)$ provides a counterexample. From above, the input "starts" at $t = 1$, while the output "starts earlier" at $t = -1$.

PROBLEM 9.8:

$$y(t) = e^{-at}u(t) * e^{-at}u(t)$$

$$= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)\, e^{-a(t-\tau)}u(t-\tau)\, d\tau$$

$$= \int_{0}^{t} e^{-a\tau} e^{-at} e^{a\tau}\, d\tau \qquad \underline{\underline{\text{If}}}\ t \geq 0$$

$$= \int_{0}^{t} e^{-at}\, d\tau = e^{-at}\int_{0}^{t} d\tau = t e^{-at}$$

If $t<0$, then $u(t)\,u(t-\tau)=0$ and the integrand is zero. So we can write the final answer as:

$$y(t) = t e^{-at} u(t)$$

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PROBLEM 9.12:

$$h(t) = e^{-0.1(t-2)}\left(u(t-2) - u(t-12) \right)$$

(a) The system is stable because $\int |h(t)| dt < \infty$

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{2}^{12} |e^{-0.1(t-2)}| dt < \int_{2}^{12} dt = 10 < \infty$$

(b) The system is causal because $h(t) = 0$ for $t < 0$.

A plot of $h(t)$ starts at $t = 2$.

(c) $x(t) = \delta(t-2)$

$$\Rightarrow y(t) = \delta(t-2) * h(t)$$

$$= h(t-2)$$

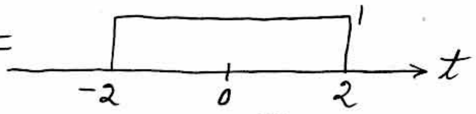
$$= e^{-0.1(t-4)}\left(u(t-4) - u(t-14) \right)$$

PROBLEM 9.17:

(a) $$y(t) = \int_{t-2}^{t+2} x(\tau)\, d\tau$$

let $x(\tau) = \delta(\tau)$ $\quad h(t) = \int_{t-2}^{t+2} \delta(\tau)\, d\tau = u(\tau)\Big|_{t-2}^{t+2}$

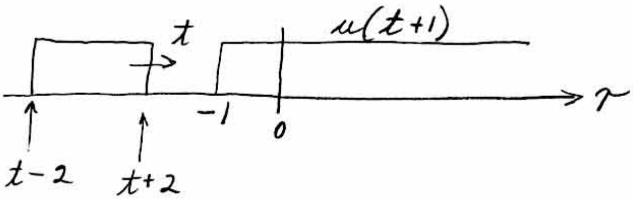
$$= u(t+2) - u(t-2) =$$



(b) Yes, it is stable because $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

(c) No, it is not causal because $h(t) = 1$ for $-2 < t < 0$

(d) $$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} u(t+1)\left[u(t-\tau+2) - u(t-\tau-2)\right] d\tau$$



Region 1 $\quad t+2 < -1 \quad t < -3 \quad y(t) = 0$

Region 2 $\quad t+2 \geq -1$ and $t-2 < -1 \quad 1 > t \geq -3$

$$y(t) = \int_{-1}^{t+2} 1 \cdot d\tau = \tau\Big|_{-1}^{t-2} = t+2+1 = t+3$$

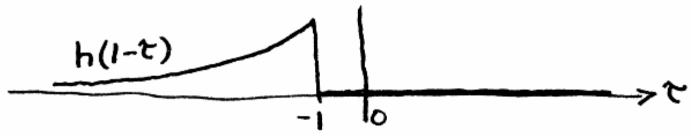
Region 3 $\quad t-2 > -1 \quad t > 1 \quad y(t) = \int_{t-2}^{t+2} 1 \cdot d\tau = 4$

PROBLEM 9.20:

(a) $h(t-\tau)$ for $t=1$ is $h(1-\tau) = e^{-(1-\tau-2)}u(1-\tau-2)$

FLIP & SHIFT by 1

$h(1-\tau) = e^{-(-1-\tau)}u(-1-\tau)$ STARTS @ $\tau=-1$



(b) Yes, the system is causal because $h(t)=0$ for $t<0$.
In fact, $h(t)=0$ for $t<2$.

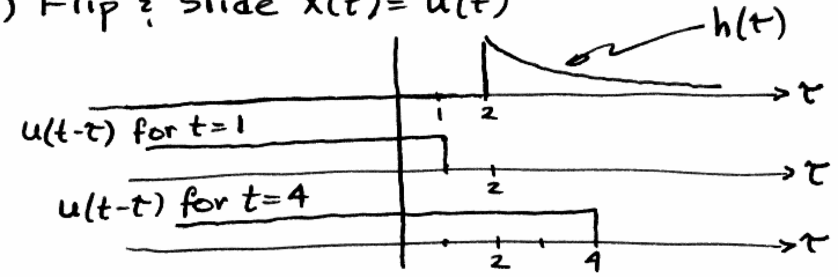
(c) To test for stability we do the integral $\int_{-\infty}^{\infty}|h(t)|dt$

$$\int_{-\infty}^{\infty}|e^{-(t-2)}u(t-2)|dt = \int_{2}^{\infty}e^{-(t-2)}dt = \left.\frac{e^{-(t-2)}}{-1}\right|_{2}^{\infty} = 0 - \frac{e^{0}}{-1} = 1 < \infty$$

Thus the system is stable.

(d) See the result from the convolution below: $t_1 = 2$

(e) Flip & Slide $x(t) = u(t)$



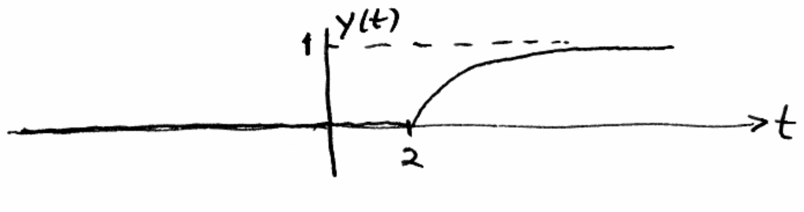
From the drawings, there is NO overlap when $t<2$.
$\Rightarrow y(t)=0$ for $t<2$.

For $t \geq 2$, we have overlap from $\tau=2$ up to $\tau=t$.

$$y(t) = \int_{2}^{t} 1\cdot e^{-(\tau-2)}d\tau = \left.\frac{e^{-(\tau-2)}}{-1}\right|_{2}^{t}$$

$$y(t) = \frac{e^{-(t-2)}}{-1} - \frac{e^{0}}{-1} = 1 - e^{-(t-2)}$$

$$\therefore y(t) = (1-e^{-(t-2)})u(t-2)$$



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PROBLEM 9.24:

(a) when $x(t) = \delta(t)$

$w_1(t) = \delta(t+1)$ and $w_2(t) = \delta(t-2)$

$\Rightarrow v(t) = w_1(t) - w_2(t) = \delta(t+1) - \delta(t-2)$

The impulse response of an integrator is $u(t)$

$\Rightarrow y(t) = u(t) * [\delta(t+1) - \delta(t-2)]$

$= u(t+1) - u(t-2)$ ← This is $h(t)$

$$h(t) = u(t+1) - u(t-2) = \begin{cases} 1 & -1 \le t < 2 \\ 0 & \text{elsewhere} \end{cases}$$

(b) The overall system is <u>NOT</u> causal

Because $h(t) \neq 0$ for $t < 0$

(c) The overall system is stable

Because $\int_{-\infty}^{\infty} |h(t)| dt = \int_{-1}^{2} 1 \, dt = 3 < \infty$