

Signal Processing First

Lecture 7 Fourier Series & Spectrum

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READING ASSIGNMENTS

- This Lecture:
 - **Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6**
 - Replaces pp. 62-66 in Ch 3 in DSP First
 - Notation: a_k for Fourier Series
- Other Reading:
 - Next Lecture: Sampling

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LECTURE OBJECTIVES

- **ANALYSIS** via Fourier Series
 - For **PERIODIC** signals: $x(t+T_0) = x(t)$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k/T_0)t} dt$$

- **SPECTRUM** from Fourier Series
 - a_k is Complex Amplitude for k-th Harmonic

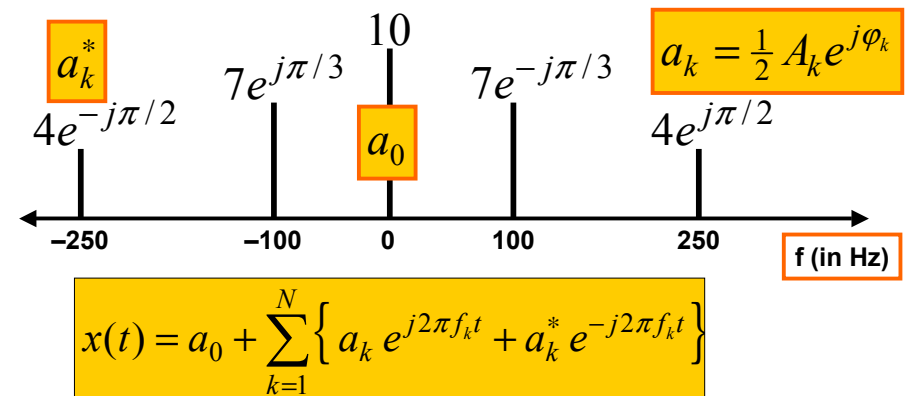
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SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



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Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

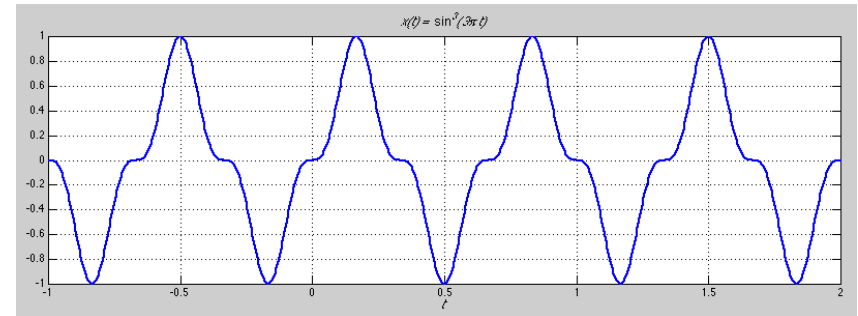
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Example

$$x(t) = \sin^3(3\pi t)$$



$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

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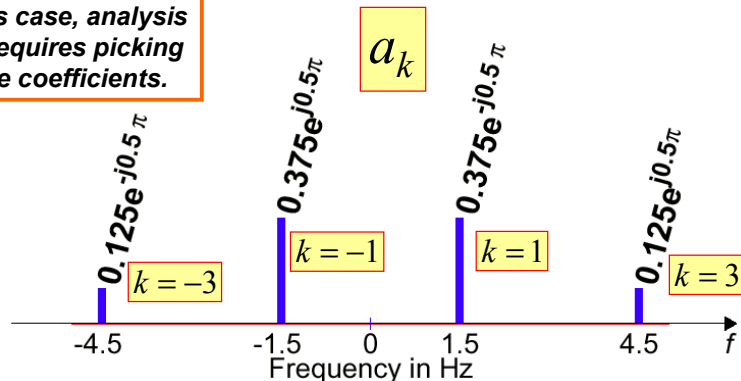
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Example

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In this case, analysis just requires picking off the coefficients.



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STRATEGY: $x(t) \rightarrow a_k$

ANALYSIS

- Get representation from the signal
- Works for **PERIODIC** Signals
- Fourier Series
 - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

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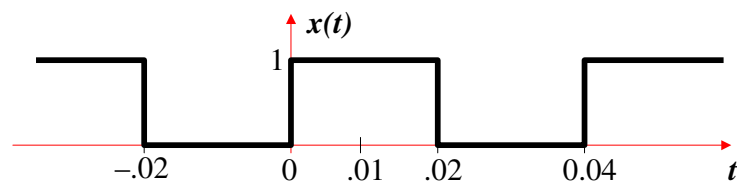
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SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} T_0 \\ 0 & \frac{1}{2} T_0 \leq t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



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FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k \neq 0)$$

$$a_k = \frac{1}{.04} \int_0^{.02} 1 e^{-j(2\pi/.04)kt} dt = \frac{1}{.04(-j2\pi k/.04)} e^{-j(2\pi/.04)kt} \Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)} (e^{-j(\pi)k} - 1) = \frac{1 - (-1)^k}{j2\pi k}$$

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DC Coefficient: a_0

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k = 0)$$

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt = \frac{1}{T_0} (\text{Area})$$

$$a_0 = \frac{1}{.04} \int_0^{.02} 1 dt = \frac{1}{.04} (.02 - 0) = \frac{1}{2}$$

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Fourier Coefficients a_k

- a_k is a function of k
 - Complex Amplitude for k -th Harmonic
 - This one doesn't depend on the period, T_0

$$a_k = \frac{1 - (-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

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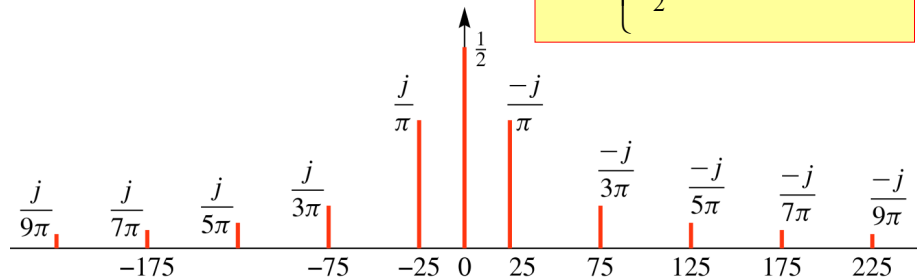
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Spectrum from Fourier Series

$$\omega_0 = 2\pi / (0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$



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Fourier Series Synthesis

- HOW do you **APPROXIMATE** $x(t)$?

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

- Use **FINITE** number of coefficients

$$x(t) = \sum_{k=-N}^N a_k e^{j2\pi k f_0 t}$$

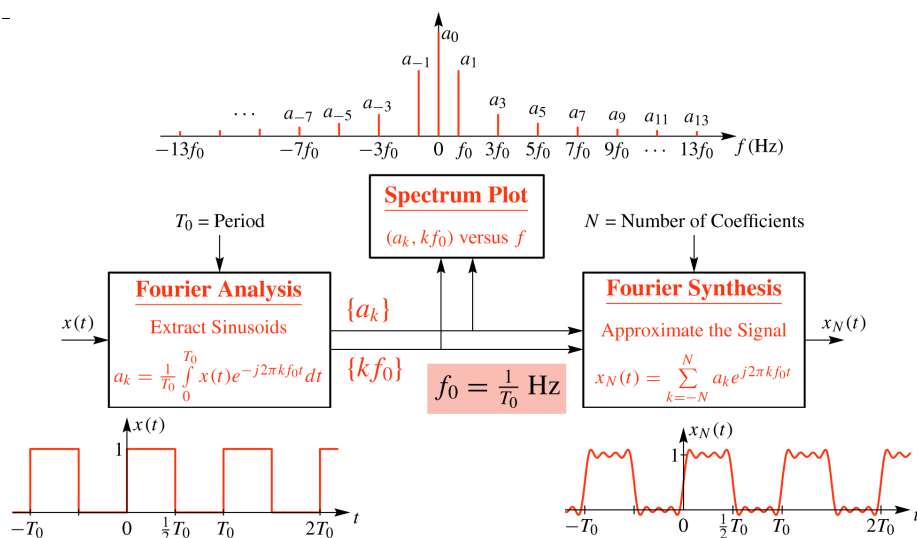
$$a_{-k} = a_k^* \text{ when } x(t) \text{ is real}$$

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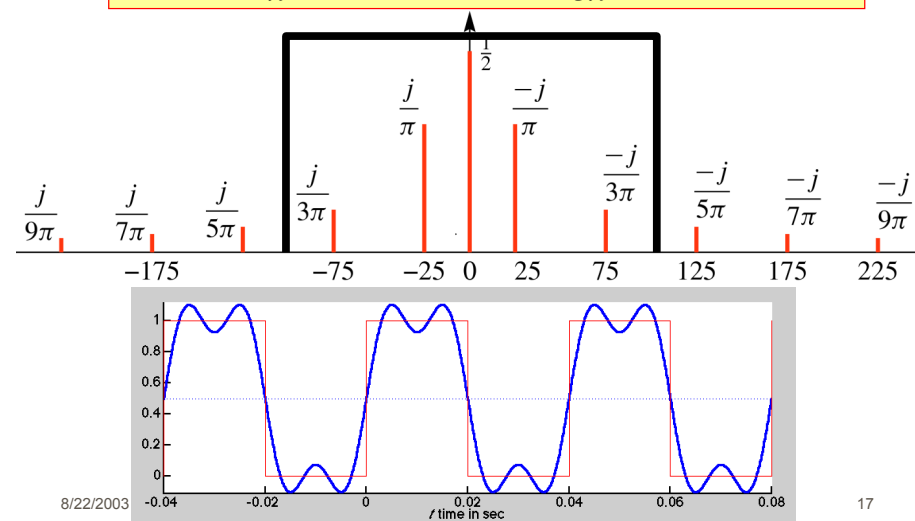
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Fourier Series Synthesis



Synthesis: 1st & 3rd Harmonics

$$y(t) = \frac{1}{2} + \frac{2}{\pi} \cos(2\pi(25)t - \frac{\pi}{2}) + \frac{2}{3\pi} \cos(2\pi(75)t - \frac{\pi}{2})$$

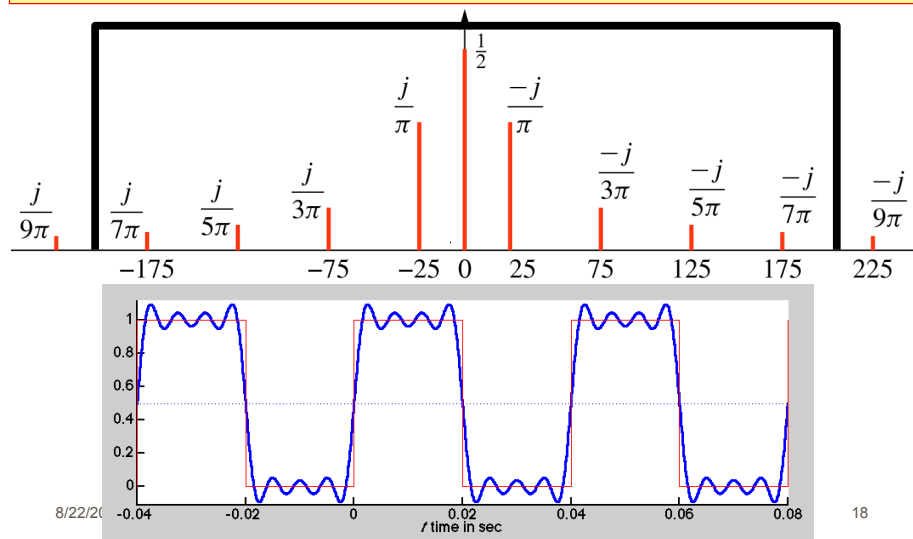


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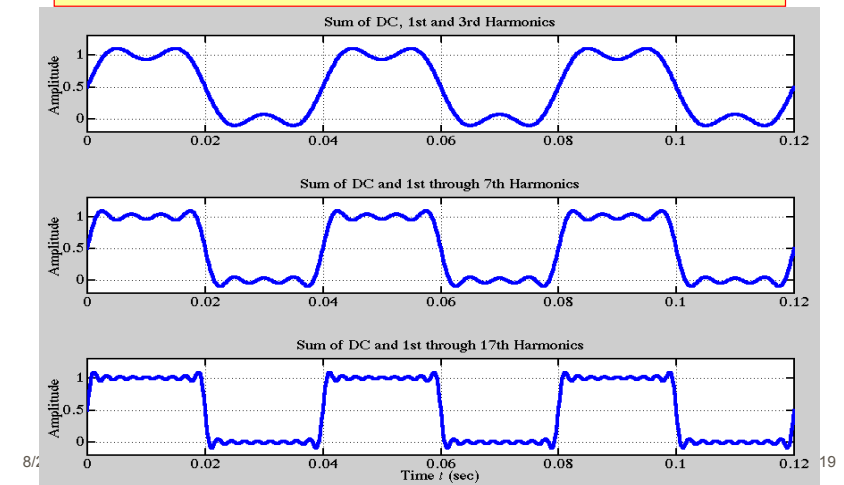
Synthesis: up to 7th Harmonic

$$y(t) = \frac{1}{2} + \frac{2}{\pi} \cos(50\pi t - \frac{\pi}{2}) + \frac{2}{3\pi} \sin(150\pi t) + \frac{2}{5\pi} \sin(250\pi t) + \frac{2}{7\pi} \sin(350\pi t)$$



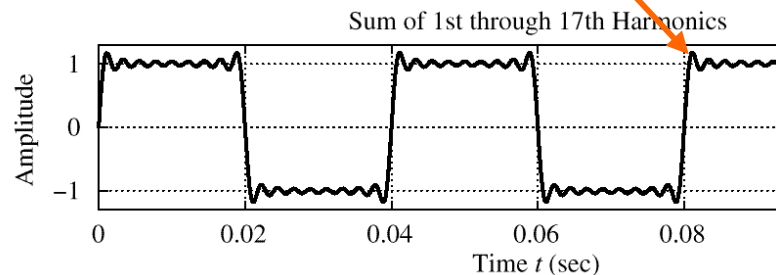
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi} \sin(\omega_0 t) + \frac{2}{3\pi} \sin(3\omega_0 t) + \dots$$



Gibbs' Phenomenon

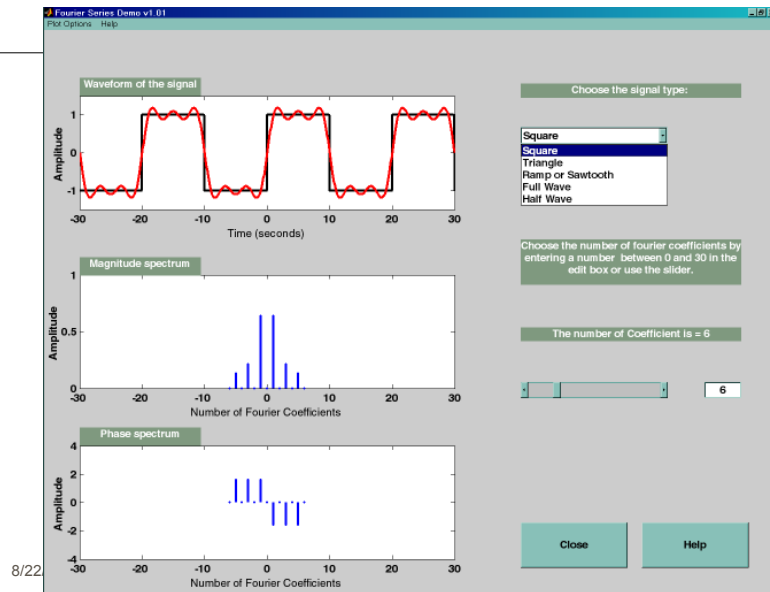
- Convergence at **DISCONTINUITY** of $x(t)$
 - There is always an **overshoot**
 - 9%** for the Square Wave case



Fourier Series Demos

- Fourier Series Java Applet
 - Greg Slabaugh
 - Interactive
 - <http://users.ece.gatech.edu/~slabaugh/java/fourier/fourier.html>
- MATLAB GUI: fseriesdemo
 - <http://users.ece.gatech.edu/mcclella/matlabGUIs/index.html>

fseriesdemo GUI



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