Fundamentals of DSP
Chap 2: Sinusoids

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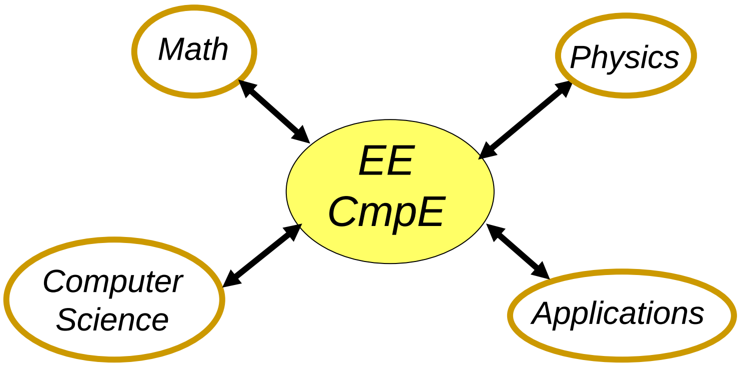
Who Should Take this Course?

- Applications of DSP
  - Multimedia (audio, speech, image, and video) signal processing
  - Communication and Networking
  - Biomedical applications
  - Radar
  - Seismic wave analysis
  - SOC for signal processing and communication
  - Time series analysis (e.g., power load forecasting, Stock market trend analysis, etc.)

Covering Fields



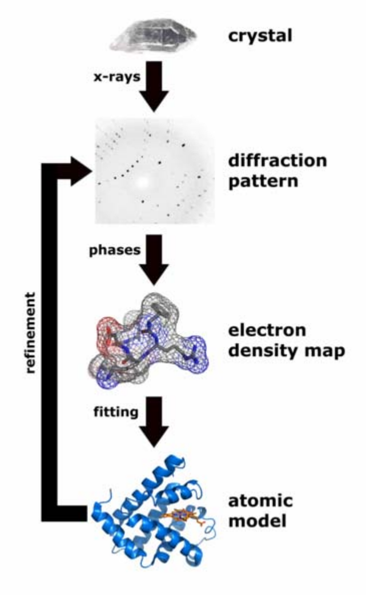
Why to Use DSP?

- Mathematical **abstractions** lead to generalization and discovery of new processing techniques
- Computer implementations are **flexible**
- Applications provide a **physical** context

Where to Use DSP?

- Telecommunications
- **Sound & Music**
  - CDROM, Digital Video
- Fourier Optics
- X-ray Crystallography
  - Protein Structure & DNA
- Computerized Tomography
- Nuclear Magnetic Resonance: MRI
- Radioastronomy
- Ref: Prestini, “The Evolution of Applied Harmonic Analysis”

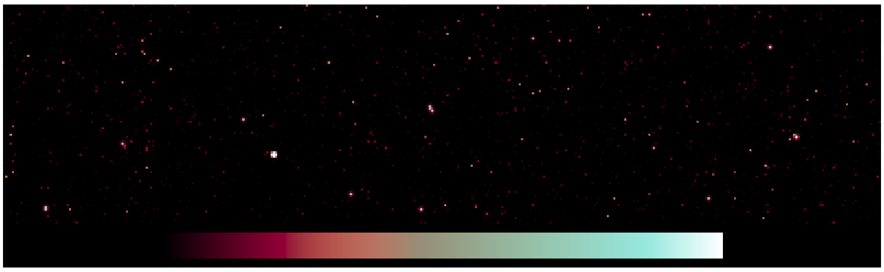
X-Ray Crystallography



Computerized Tomography

MRI Image of Brain

Radioastronomy



Chapter Objectives

- Write general formula for a “sinusoidal” waveform, or signal
- From the formula, plot the sinusoid versus time
- Define sinusoidal from a plot
- Relate Time-Shift to Phase

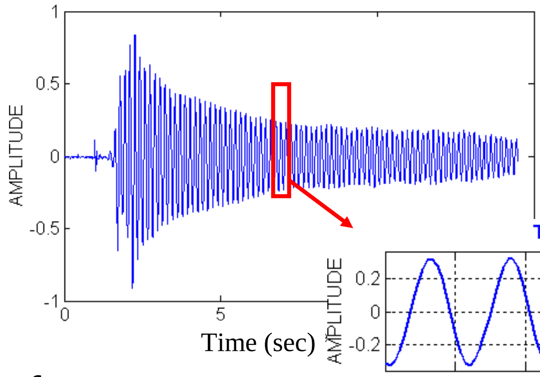
- What’s a **signal**?
  - It’s a **function** of time, *x*(*t*)
  - in the mathematical sense

Tuning Fork Example

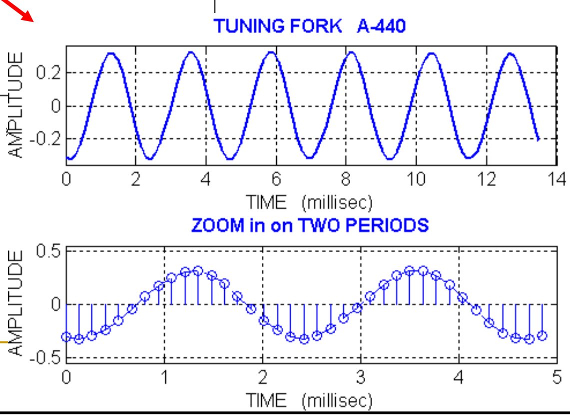
- CD-ROM demo
- "A" is at 440 Hertz (Hz)
- Waveform is a SINUSOIDAL SIGNAL
- Computer plot looks like a sine wave
- This should be the mathematical formula:

$$A\cos(2\pi(440)t+\varphi)$$

Tuning Fork A-440 Waveform



$$T \approx 8.15 - 5.85$$
$$= 2.3\,\text{ms}$$



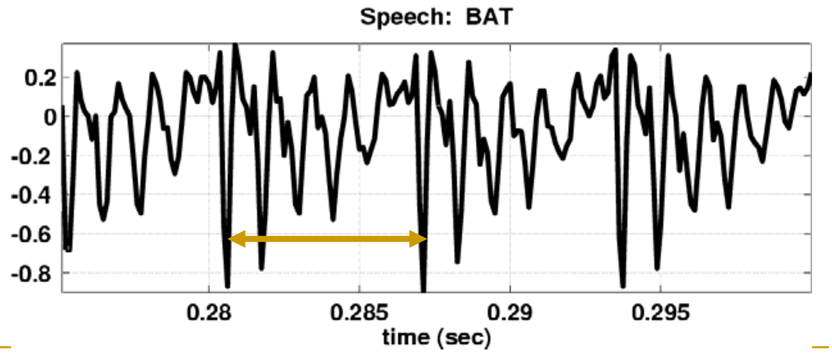
$$f = 1/T$$
$$= 1000/2.3$$
$$\approx 435\,\text{Hz}$$

Speech Example

- More complicated signal (BAT.WAV)
- Waveform ***x*(*t*)** is NOT a Sinusoid
- Theory will tell us
  - ***x*(*t*)** is approximately a sum of sinusoids
  - FOURIER ANALYSIS
    - Break ***x*(*t*)** into its sinusoidal components
  - Called the FREQUENCY SPECTRUM

Speech Signal: BAT

- Nearly **Periodic** in Vowel Region
  - Period is (Approximately) T = 0.0065 sec



Digitizing the Waveform

- ***x*[*n*]** is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- Sample at 11,025 samples per second
  - Called the SAMPLING RATE of the A/D
  - Time between samples is
    - 1/11025 = 90.7 microsec
- Output via D/A hardware (at F_{samp})

Storing Digital Sound

- ***x*[*n*]** is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- CD rate is 44,100 samples per second
- 16-bit samples
- Stereo uses 2 channels
- Number of bytes for 1 minute is
  - 2 X (16/8) X 60 X 44100 = 10.584 Mbytes

Sines and Cosines

- Always use the COSINE FORM

$$A\cos(2\pi(440)t + \varphi)$$

- Sine is a special case:

$$\sin(\omega t) = \cos(\omega t - \tfrac{\pi}{2})$$

Sinusoidal Signal

$$A\cos(\omega t + \varphi)$$

- **FREQUENCY** ω
  - Radians/sec
  - Hertz (cycles/sec)

$$\omega = (2\pi) f$$

- **PERIOD** (in sec)

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

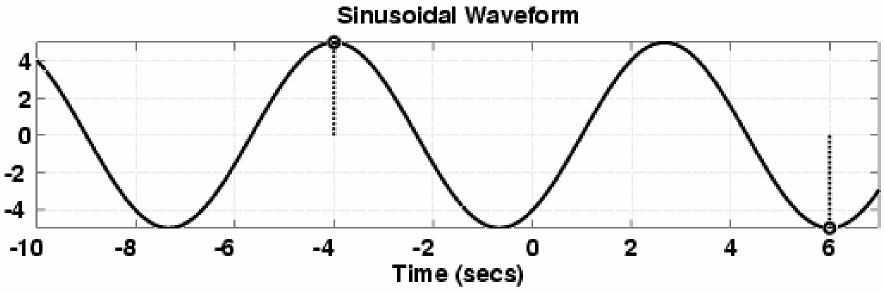
- **AMPLITUDE** A
  - Magnitude
- **PHASE** φ

Example of Sinusoid

- Given the Formula

$$5\cos(0.3\pi t + 1.2\pi)$$

- Make a plot



Plot a Cosine Signal

$$5\cos(0.3\pi t + 1.2\pi)$$

- Formula defines *A*, ω, and φ

$$A = 5$$
$$\omega = 0.3\pi$$
$$\varphi = 1.2\pi$$

Plot a Cosine Signal from the Formula

$$5\cos(0.3\pi t + 1.2\pi)$$

- Determine **period**:

$$T = 2\pi/\omega = 2\pi/0.3\pi = 20/3$$

- Determine a **peak** location by solving

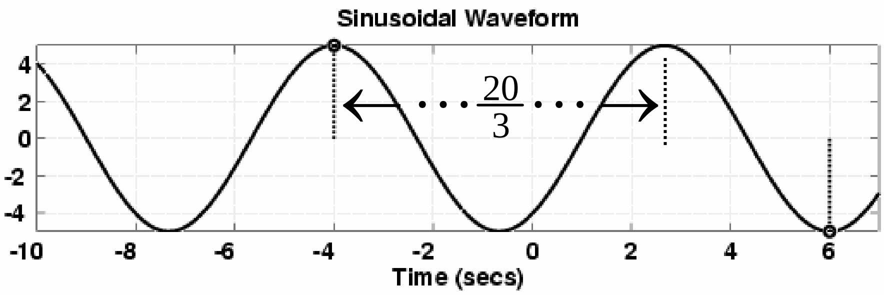
$$(\omega t + \varphi) = 0 \quad \Rightarrow (0.3\pi t + 1.2\pi) = 0$$

- **Zero** crossing is *T*/4 before or after
- Positive & Negative peaks spaced by *T*/2

Plot the Sinusoid

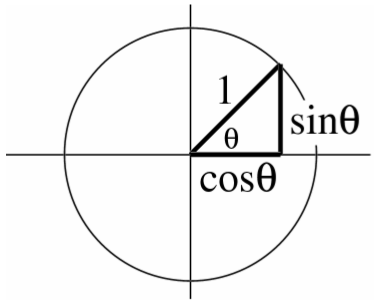
$$5\cos(0.3\pi t + 1.2\pi)$$

- Use *T* = 20/3 and the peak location at *t* = -4



Trig Functions

- Circular Functions



- Common Values
  - $\sin(k\pi) = 0$
  - $\cos(0) = 1$
  - $\cos(2k\pi) = 1$ and $\cos((2k+1)\pi) = -1$
  - $\cos((k+0.5)\pi) = 0$

Time Shift

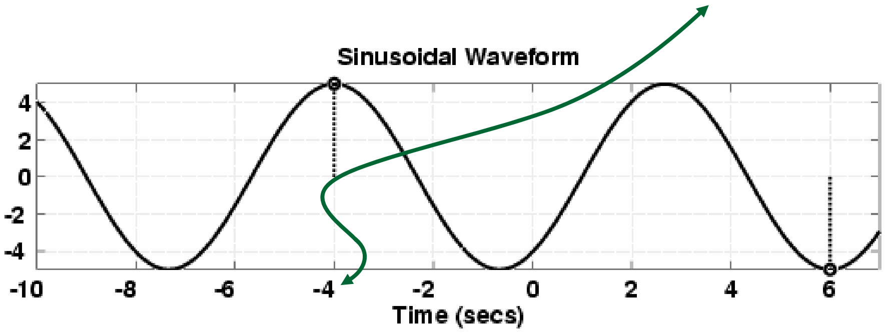
- In a mathematical formula we can replace t with $t - t_m$

$$x(t - t_m) = A\cos(\omega(t - t_m))$$

- Then the $t = 0$ point moves to $t = t_m$
- Peak value of $\cos(\omega(t - t_m))$ is now at $t = t_m$

Time-Shifted Sinusoidal

$$x(t+4) = 5\cos(0.3\pi(t+4)) = 5\cos(0.3\pi(t-(-4)))$$



Phase ↔ Time-Shift

- Equate the formulas:

$$A\cos(\omega(t-t_m)) = A\cos(\omega t+\varphi)$$

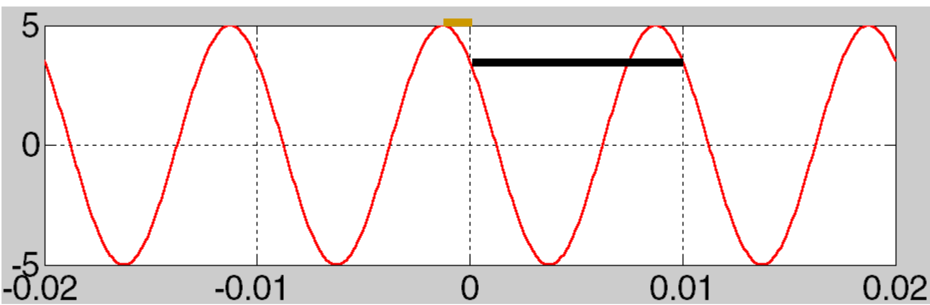
- and we obtain: $-\omega t_m = \varphi$
- or, $t_m = -\dfrac{\varphi}{\omega}$

Sinusoid from a Plot

- Measure the period, *T*
  - Between peaks or zero crossings
  - Compute frequency: $\omega = 2p/T$
- Measure time of a peak: t_m
  - Compute phase: $f = -\,\omega\, t_m$
- Measure height of positive peak: *A*

3 steps

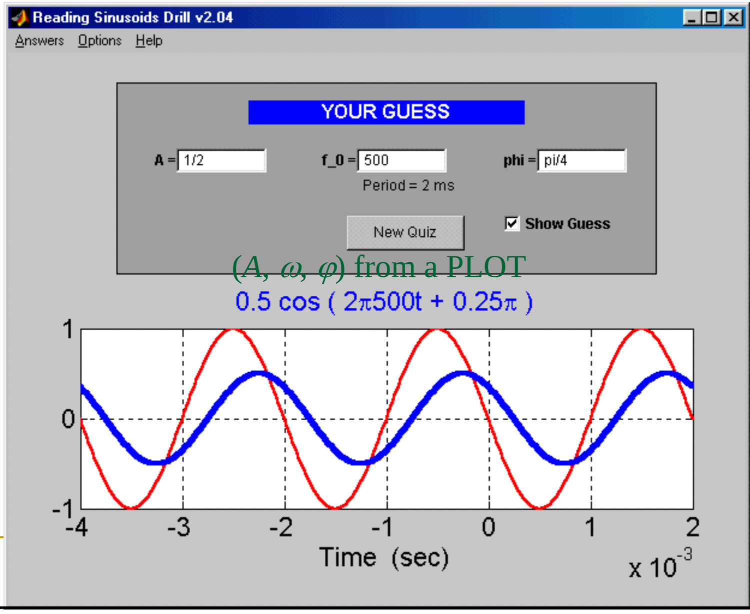
(A, ω, φ) from a PLOT



$$T = \frac{0.01\text{sec}}{1\,\text{period}} = \frac{1}{100} \;\longrightarrow\; \omega = \frac{2\pi}{T} = \frac{2\pi}{0.01} = 200\pi$$

$$t_m = -0.00125\text{sec} \;\longrightarrow\; \varphi = -\omega t_m = -(200\pi)(t_m) = 0.25\pi$$

Sine Drill (MATLAB GUI)



Phase is Ambiguous

- The cosine signal is periodic
  - Period is 2π

$$A\cos(\omega t + \varphi + 2\pi) = A\cos(\omega t + \varphi)$$

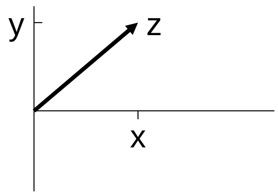
  - Thus adding any multiple of 2π leaves $x(t)$ unchanged

$$\text{if} \quad t_m = \frac{-\varphi}{\omega}, \quad \text{then}$$

$$t_{m_2} = \frac{-(\varphi + 2\pi)}{\omega} = \frac{-\varphi}{\omega} - \frac{2\pi}{\omega} = t_m - T$$

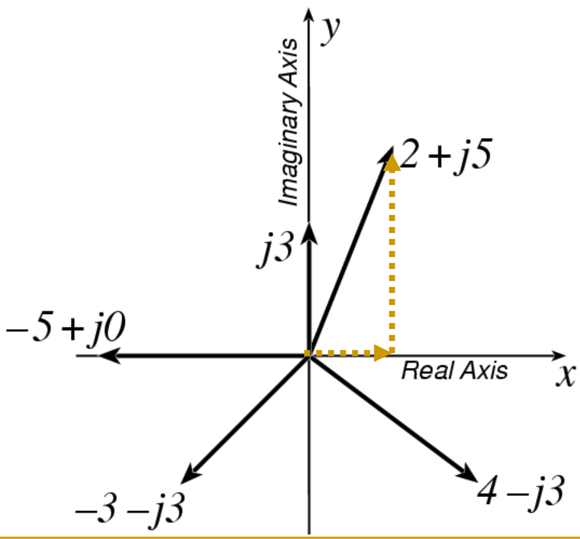
Complex Numbers

- To solve: $z^2 = -1$
  - $z = \boldsymbol{j}$
  - Math and Physics use $z = \boldsymbol{i}$
- Complex number: $z = x + \boldsymbol{j}\, y$

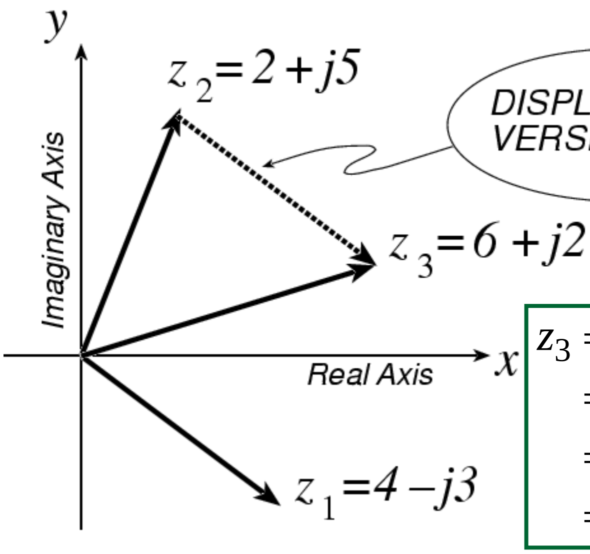


Cartesian coordinate system

Plot Complex Numbers



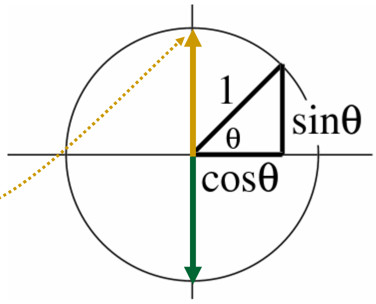
Complex Addition = Vector Addition



$$
\begin{aligned}
z_3 &= z_1 + z_2 \\
&= (4 - j3) + (2 + j5) \\
&= (4 + 2) + j(-3 + 5) \\
&= 6 + j2
\end{aligned}
$$

Polar Form

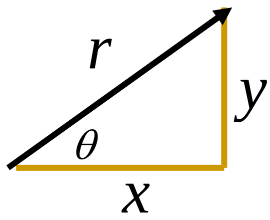
- Vector Form
  - **Length** =1
  - **Angle** = θ
- Common Values
  - ***j*** has angle of 0.5π
  - -1 has angle of π
  - **-** ***j*** has angle of 1.5π
  - also, angle of ***-j*** **could** be $-0.5\pi = 1.5\pi - 2\pi$
  - because the PHASE is **AMBIGUOUS**



Polar ↔ Rectangular

- Relate (*x*,*y*) to (*r*,*q*)

$$r^2 = x^2 + y^2$$
$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$$



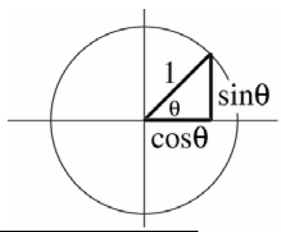
Most calculators do Polar-Rectangular

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Need a notation for POLAR FORM

Euler's Formula

- **Complex Exponential**
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



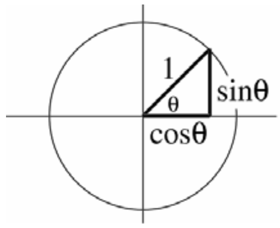
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

Complex Exponential

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Interpret this as a **Rotating Vector**
  - $\theta = \omega t$
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Cosine = Real Part

Real Part of Euler's

$$\cos(\omega t) = \Re e\{e^{j\omega t}\}$$

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi)$$

So,

$$A\cos(\omega t + \varphi) = \Re e\{Ae^{j(\omega t + \varphi)}\}$$
$$= \Re e\{Ae^{j\varphi}e^{j\omega t}\}$$

Real-Part Example

$$A\cos(\omega t+\varphi)=\Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Evaluate: $$x(t)=\Re e\left\{-3je^{j\omega t}\right\}$$

Answer:

$$x(t)=\Re e\left\{(-3j)e^{j\omega t}\right\}$$
$$=\Re e\left\{3e^{-j0.5\pi}e^{j\omega t}\right\}=3\cos(\omega t-0.5\pi)$$

Complex Amplitude

General Sinusoid

$$x(t)=A\cos(\omega t+\varphi)=\Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Complex AMPLITUDE = X

$$z(t)=Xe^{j\omega t} \qquad X=Ae^{j\varphi}$$

Then, any Sinusoid = REAL PART of $Xe^{j\omega t}$

$$x(t)=\Re e\left\{Xe^{j\omega t}\right\}=\Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

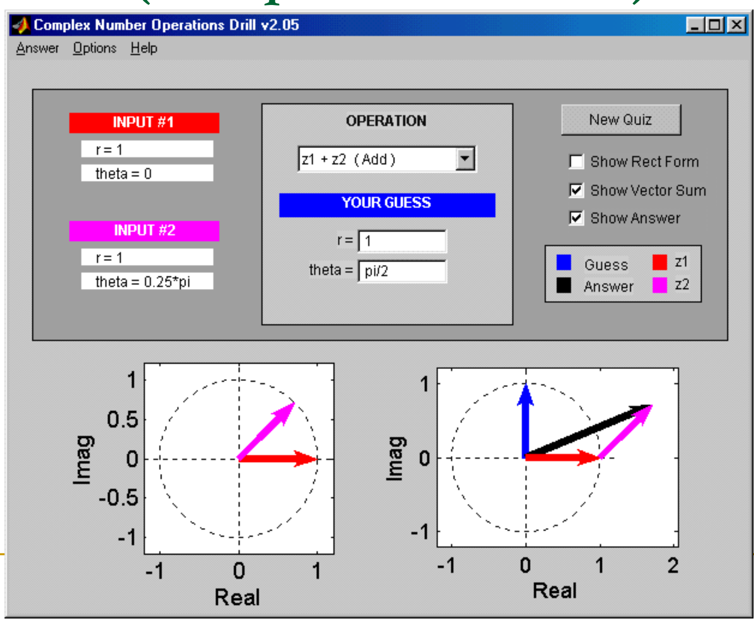
Phasor Representation

- Phasors = Complex Amplitude
  - Complex Numbers **represent** Sinusoids

$$z(t) = Xe^{j\omega t} = (Ae^{j\varphi})e^{j\omega t}$$

- Develop the ABSTRACTION:
  - Adding Sinusoids = Complex Addition
  - PHASOR ADDITION THEOREM

Z Drill (Complex Arithmetic)



AVOID Trigonometry

- Algebra, even complex, is EASIER !!!
- Can you recall $\cos(\theta_1 + \theta_2)$?
- Use: real part of $e^{j(\theta_1+\theta_2)} = \cos(\theta_1 + \theta_2)$

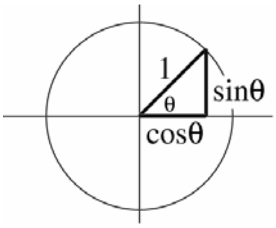
$$e^{j(\theta_1+\theta_2)} = e^{j\theta_1} e^{j\theta_2}$$

$$= (\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2)$$

$$= \boxed{(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)} + j(\ldots)$$

Euler's Formula

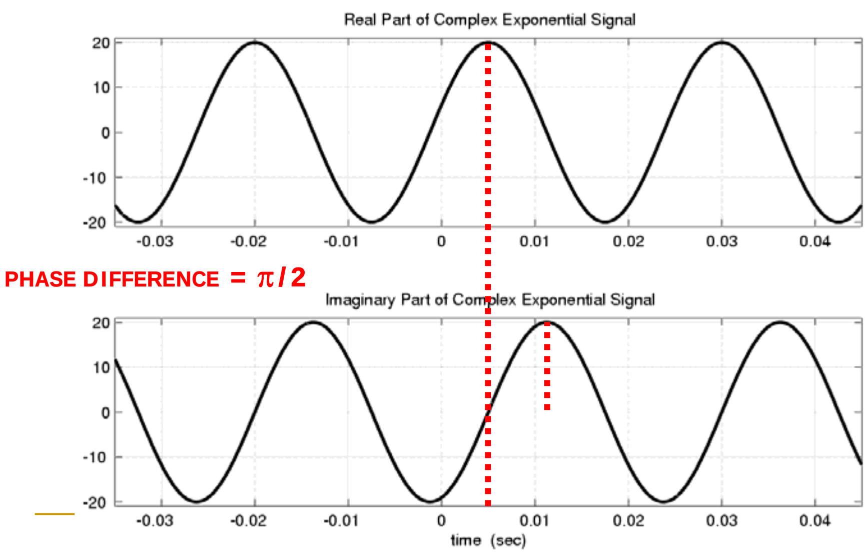
- Complex Exponential
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\boxed{e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)}$$

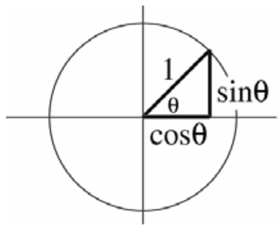
Real & Imaginary Part Plots



Complex Exponential

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

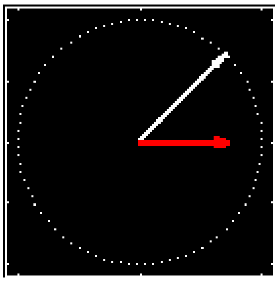
- Interpret this as a **Rotating Vector**
  - $\theta = \omega t$
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates $0.2\,\pi$ in 0.01 s

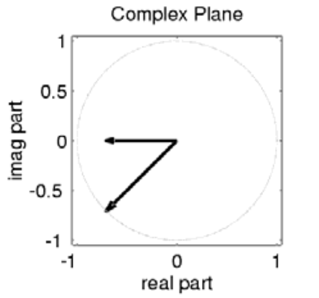


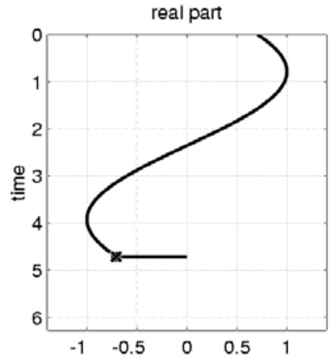
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Rotating Phasor

See Demo on CD-ROM
Chapter 2







Addition of Two Sinusoids

- ALL SINUSOIDS have SAME FREQUENCY
- HOW to GET {Amp,Phase} of RESULT ?

$$x_1(t) = 1.7\cos(2\pi(10)t + 70\pi/180)$$

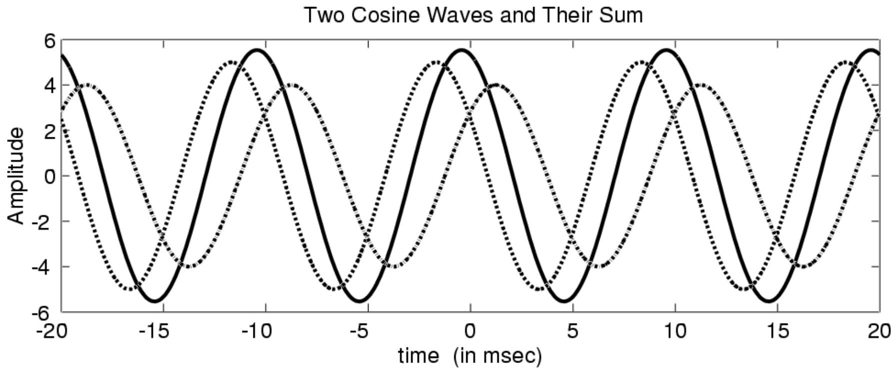
$$x_2(t) = 1.9\cos(2\pi(10)t + 200\pi/180)$$

$$x_3(t) = x_1(t) + x_2(t)$$

$$= 1.532\cos(2\pi(10)t + 141.79\pi/180)$$

Addition of Two Sinusoids (Cont.)

- Sum Sinusoid has **SAME** Frequency



Phasor Addition Rule

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k)$$

$$= A\cos(\omega_0 t + \phi)$$

Get the new complex amplitude by complex addition

$$\sum_{k=1}^{N} A_k e^{j\phi_k} = Ae^{j\phi}$$

Phasor Addition Proof

$$\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = \sum_{k=1}^{N} \Re e\left\{A_k e^{j(\omega_0 t + \phi_k)}\right\}$$

$$= \Re e\left\{\sum_{k=1}^{N} A_k e^{j\phi_k} e^{j\omega_0 t}\right\}$$

$$= \Re e\left\{\left(\sum_{k=1}^{N} A_k e^{j\phi_k}\right) e^{j\omega_0 t}\right\}$$

$$= \Re e\left\{\left(A e^{j\phi}\right) e^{j\omega_0 t}\right\} = A\cos(\omega_0 t + \phi)$$

Pop Quiz: Phasor Addition

- ADD THESE 2 SINUSOIDS:

$$x_1(t) = \cos(77\pi t)$$

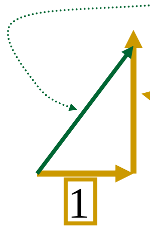
$$x_2(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)$$

- COMPLEX ADDITION:

$$1e^{j0} + \sqrt{3}e^{j0.5\pi}$$

Pop Quiz (Answer)

- COMPLEX ADDITION: $1 + j\sqrt{3} = 2e^{j\pi/3}$



$j\sqrt{3} = \sqrt{3}e^{j0.5\pi}$

- CONVERT back to cosine form:

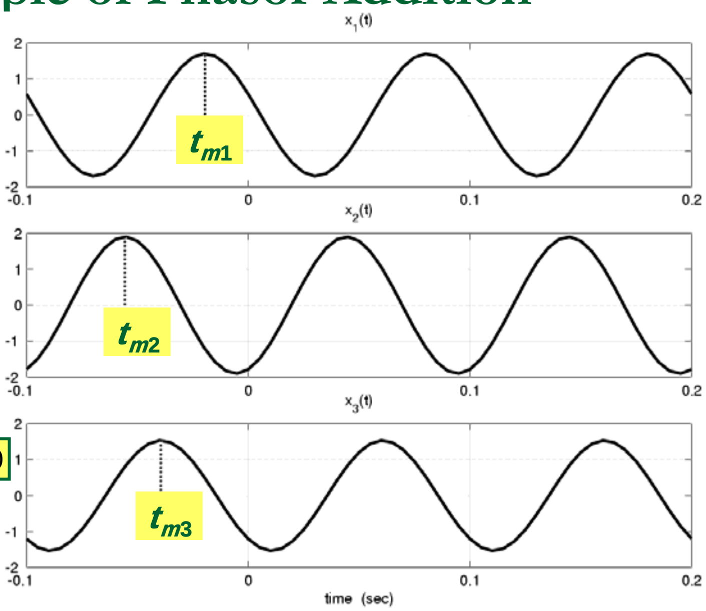
$$x_3(t) = 2\cos(77\pi t + \tfrac{\pi}{3})$$

Example of Phasor Addition

$x_1(t)$

$x_2(t)$

$x_3(t) = x_1(t) + x_2(t)$



Conversion of Time-Shift to Phase

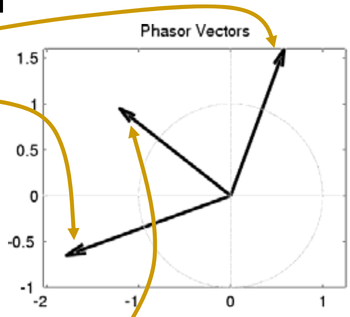
- Measure peak times:
  - t_{m1}=-0.0194, t_{m2}=-0.0556, t_{m3}=-0.0394
- Convert to phase (T=0.1)
  - $\varphi_1 = -\omega t_{m1} = -2\pi(t_{m1}/T) = 70\pi/180$,
  - φ_2 = 200p/180
- Amplitudes
  - A_1=1.7, A_2=1.9, A_3=1.532

Phasor Addition: Numerical

- Convert Polar to Cartesian
  - $X_1 = 0.5814 + j1.597$
  - $X_2 = -1.785 - j0.6498$

  sum =

  $X_3 = -1.204 + j0.9476$
- Convert back to Polar
  - X_3 = 1.532 at angle $141.79\pi/180$
  - This is the sum



Phasor Addition: Graphical

$$x_1(t) = 1.7\cos(2\pi(10)t + 70\pi/180)$$
$$x_2(t) = 1.9\cos(2\pi(10)t + 200\pi/180)$$

$$x_3(t) = x_1(t) + x_2(t)$$
$$= 1.532\cos(2\pi(10)t + 141.79\pi/180)$$

VECTOR (PHASOR) ADD