Signal Processing First

Lecture 22

Introduction to the Fourier Transform

READING ASSIGNMENTS

- This Lecture:
  - Chapter 11, Sects. 11-1 to 11-4
- Other Reading:
  - Recitation: Ch. 10
    - And Chapter 11, Sects. 11-1 to 11-4
  - Next Lecture: Chapter 11, Sects. 11-5, 11-6

LECTURE OBJECTIVES

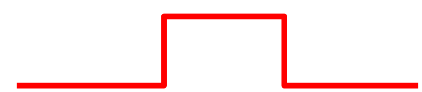
- Review
  - Frequency Response
  - Fourier Series
- Definition of Fourier transform

$$\boxed{X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt}$$

  Relation to Fourier Series
- Examples of Fourier transform pairs

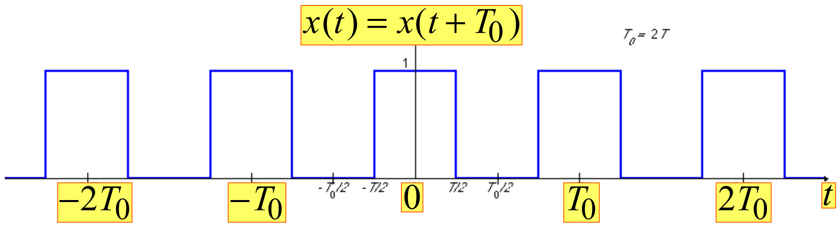
Everything = Sum of Sinusoids

- One Square Pulse = Sum of Sinusoids
  - ???????????
- Finite Length
- Not Periodic



- Limit of Square Wave as Period → infinity
  - Intuitive Argument

Fourier Series: Periodic *x(t)*



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

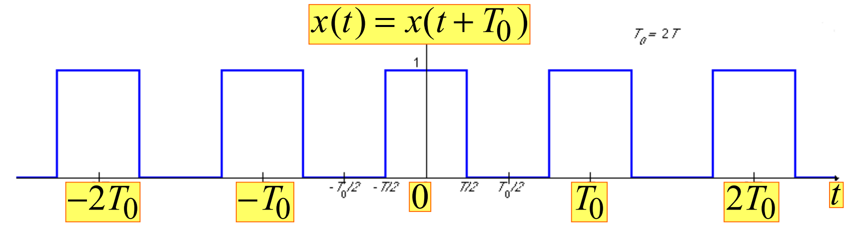
Fundamental Freq.

$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega_0 k t} dt$$

Fourier Analysis

Square Wave Signal



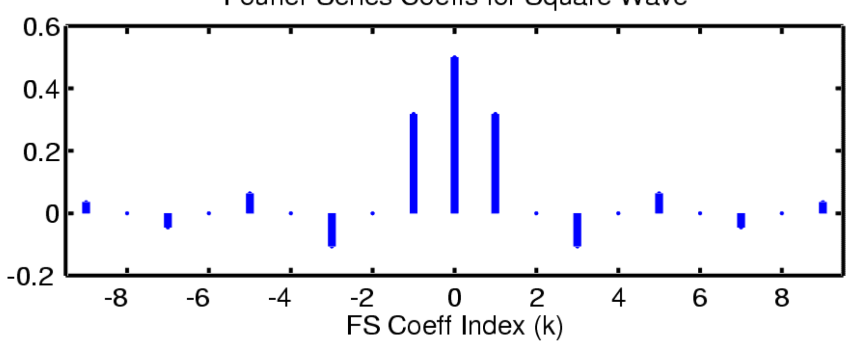
$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1) e^{-j\omega_0 k t} dt$$

$$a_k = \left. \frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0} \right|_{-T_0/4}^{T_0/4} = \frac{e^{-j\pi k/2} - e^{j\pi k/2}}{-j2\pi k} = \frac{\sin(\pi k/2)}{\pi k}$$

Spectrum from Fourier Series

$$a_k = \frac{\sin(\pi k/2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \end{cases}$$

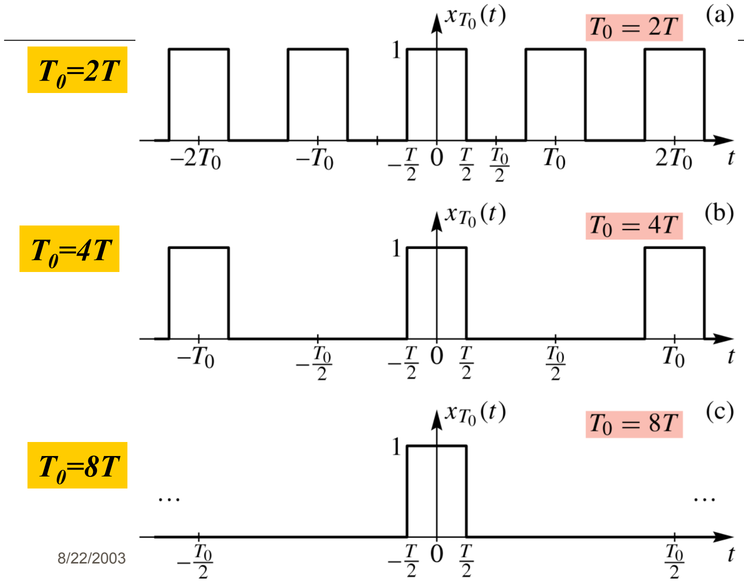
Fourier Series Coeffs for Square Wave



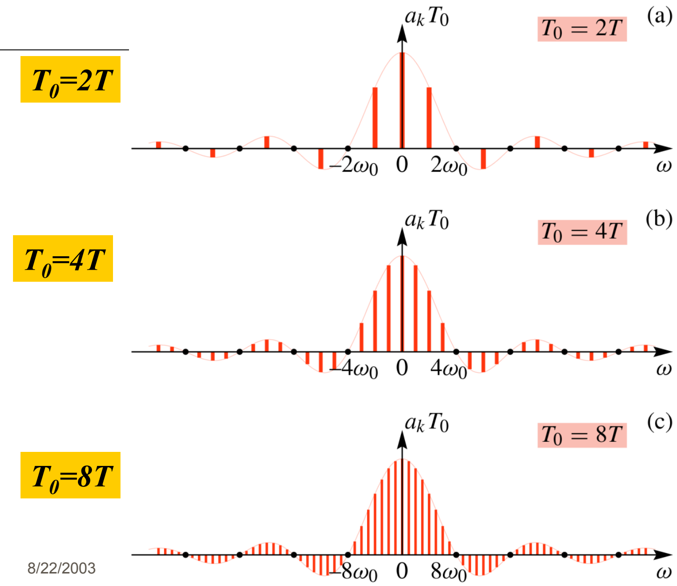
What if *x(t)* is not periodic?

- Sum of Sinusoids?
  - Non-harmonically related sinusoids
  - Would not be periodic, but would probably be non-zero for all *t.*
- Fourier transform
  - gives a "sum" (actually an **integral**) that involves **ALL** frequencies
  - can represent signals that are identically zero for negative *t*. !!!!!!!!!

Limiting Behavior of FS



Limiting Behavior of Spectrum



Plot $(T_0 a_k)$

FS in the LIMIT (long period)

$$x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}(T_0 a_k)e^{j\omega_0 kt}\left(\frac{2\pi}{T_0}\right) \mapsto x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

Fourier Synthesis

$$\lim_{T_0\to\infty}\frac{2\pi}{T_0} = d\omega \qquad \lim_{T_0\to\infty}\frac{2\pi}{T_0}k = \omega \qquad \lim_{T_0\to\infty} T_0 a_k = X(j\omega)$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t)e^{-j\omega_0 kt}\,dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Fourier Analysis

Fourier Transform Defined

- For non-periodic signals

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

Fourier Synthesis

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Fourier Analysis

Example 1: $\boxed{x(t) = e^{-at}u(t)}$

$$X(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t}\,dt = \int_0^\infty e^{-(a+j\omega)t}\,dt$$

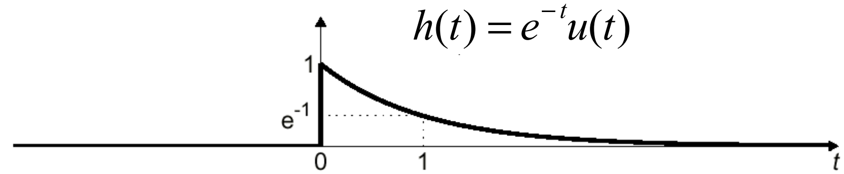
$$\boxed{a > 0}$$

$$X(j\omega) = -\frac{e^{-at}e^{-j\omega t}}{a+j\omega}\Bigg|_0^\infty = \frac{1}{a+j\omega}$$

$$\boxed{X(j\omega) = \frac{1}{a+j\omega}}$$

Frequency Response

- Fourier Transform of h(t) is the Frequency Response

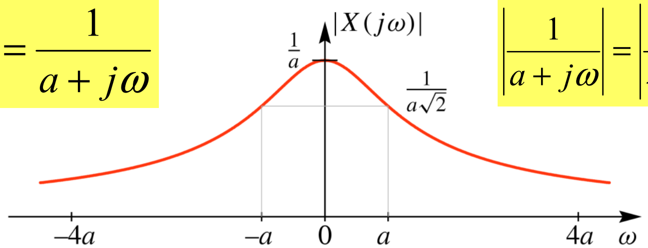


$$\boxed{h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1+j\omega}}$$

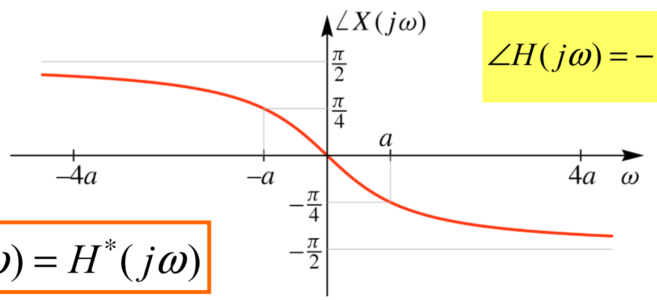
Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{a+j\omega}$$

$$\left|\frac{1}{a+j\omega}\right| = \left|\frac{1}{\sqrt{a^2+\omega^2}}\right|$$



$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



$$\boxed{H(-j\omega) = H^*(j\omega)}$$

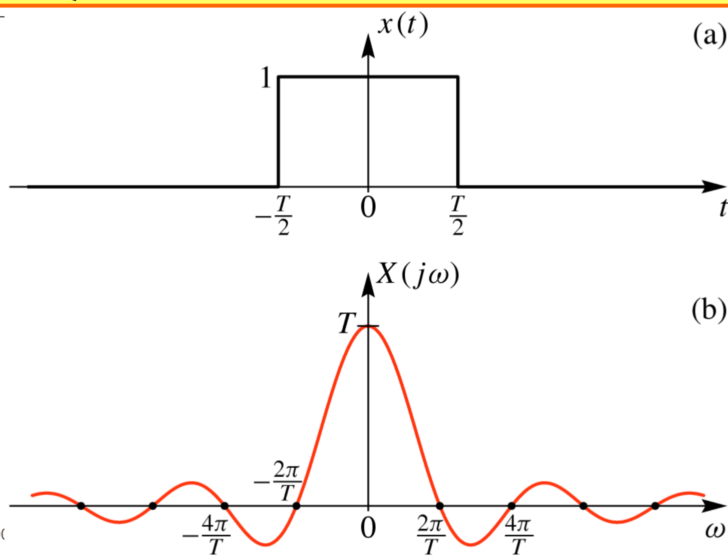
Example 2: $\boxed{x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}}$

$$X(j\omega) = \int_{-T/2}^{T/2} (1)e^{-j\omega t}\,dt = \int_{-T/2}^{T/2} e^{-j\omega t}\,dt$$

$$X(j\omega) = \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$\boxed{X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \quad \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$



Example 3:

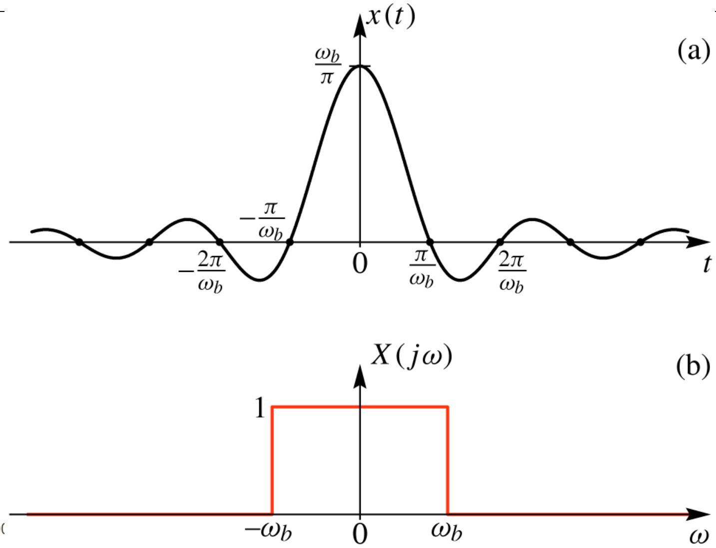
$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega t}\,d\omega$$

$$x(t) = \frac{1}{2\pi}\left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi}\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{jt}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$



Example 4:

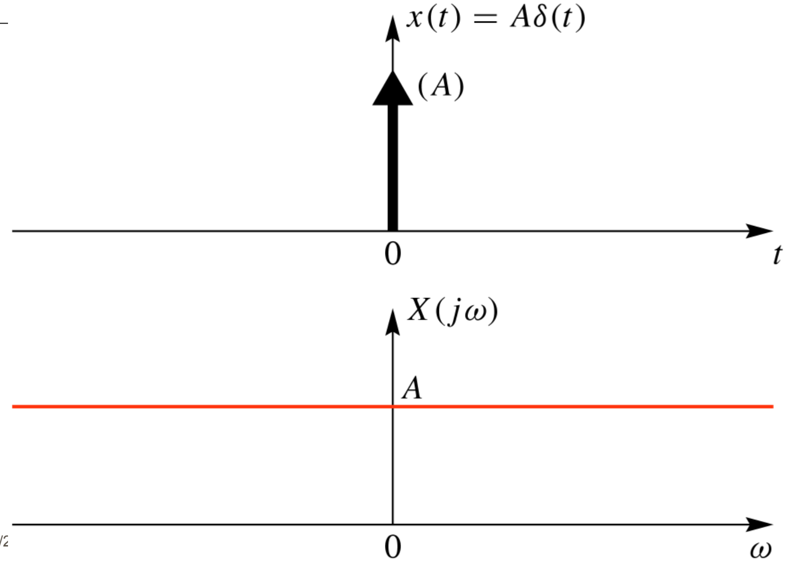
$$x(t) = \delta(t - t_0)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt = 1$$

Shifting Property of the Impulse

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t}\,dt = e^{-j\omega t_0}$$

$$x(t)=\delta(t) \Leftrightarrow X(j\omega)=1$$



Example 5: $X(j\omega)=2\pi\delta(\omega-\omega_0)$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}\,d\omega = e^{j\omega_0 t}$$

$$x(t)=e^{j\omega_0 t} \Leftrightarrow X(j\omega)=2\pi\delta(\omega-\omega_0)$$

$$x(t)=1 \Leftrightarrow X(j\omega)=2\pi\delta(\omega)$$

$$x(t)=\cos(\omega_0 t) \Leftrightarrow$$
$$X(j\omega)=\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$$

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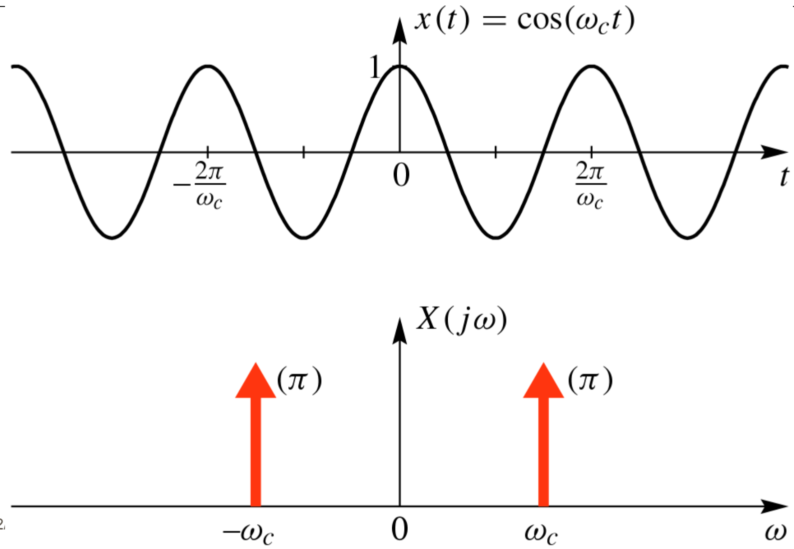

Table of Fourier Transforms

$$x(t)=e^{-at}u(t) \Leftrightarrow X(j\omega)=\frac{1}{a+j\omega}$$

$$x(t)=\begin{cases}1 & |t|<T/2\\ 0 & |t|>T/2\end{cases} \Leftrightarrow X(j\omega)=\frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t)=\frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega)=\begin{cases}1 & |\omega|<\omega_0\\ 0 & |\omega|>\omega_0\end{cases}$$

$$x(t)=\delta(t-t_0) \Leftrightarrow X(j\omega)=e^{-j\omega t_0}$$

$$x(t)=e^{j\omega_0 t} \Leftrightarrow X(j\omega)=2\pi\delta(\omega-\omega_0)$$