

# Signal Processing First

## Lecture 23 Fourier Transform Properties

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## READING ASSIGNMENTS

- This Lecture:
  - Chapter 11, Sects. 11-5 to 11-9
  - **Tables in Section 11-9**
- Other Reading:
  - Recitation: Chapter 11, Sects. 11-1 to 11-9
  - Next Lectures: Chapter 12 (Applications)

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## LECTURE OBJECTIVES

- The Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

- More examples of Fourier transform pairs
- Basic properties of Fourier transforms
  - **Convolution** property
  - **Multiplication** property

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## Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$$

*Fourier Synthesis  
(Inverse Transform)*

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

*Fourier Analysis  
(Forward Transform)*

Time - domain  $\Leftrightarrow$  Frequency - domain

$$x(t) \Leftrightarrow X(j\omega)$$

## WHY use the Fourier transform?

- Manipulate the **“Frequency Spectrum”**
- Analog Communication Systems
  - AM: Amplitude Modulation; FM
  - What are the **“Building Blocks”** ?
    - Abstract Layer**, not implementation
- Ideal Filters: mostly BPFs
- Frequency Shifters
  - aka Modulators, Mixers or Multipliers:  $x(t)p(t)$

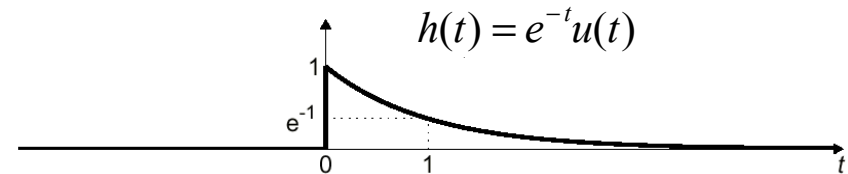
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## Frequency Response

- Fourier Transform of  $h(t)$  **is** the Frequency Response



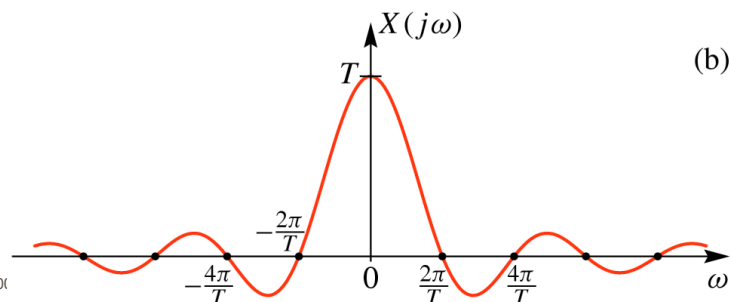
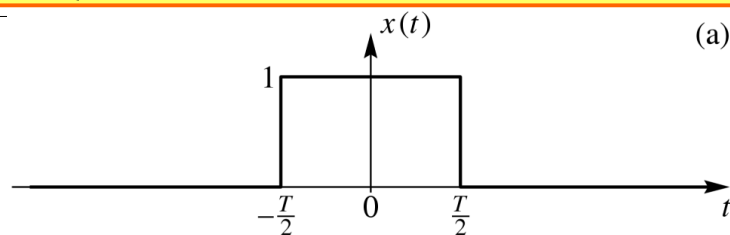
$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1 + j\omega}$$

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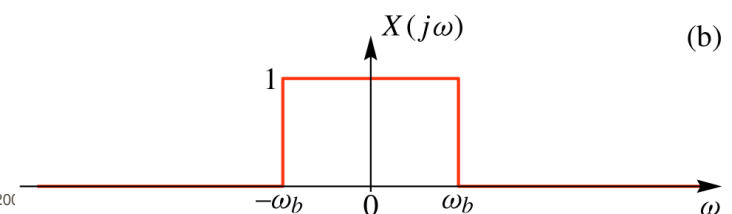
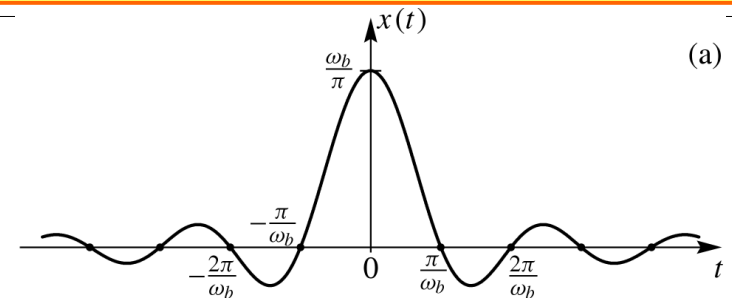
$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$



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$$x(t) = \frac{\sin(\omega_b t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$

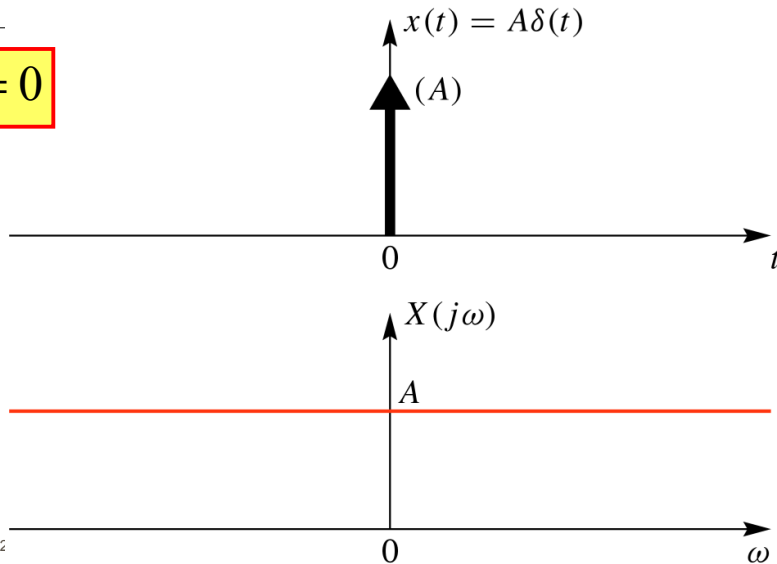


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$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$t_0 = 0$$



## Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

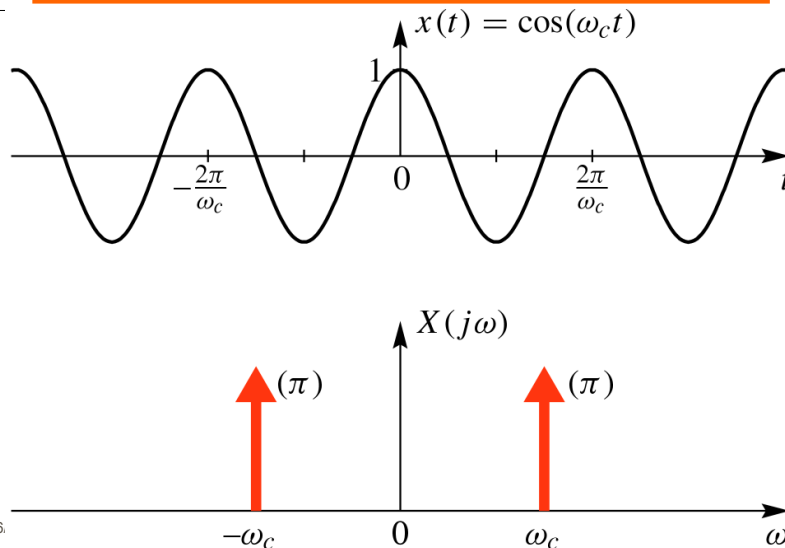
$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



## Fourier Transform of a General Periodic Signal

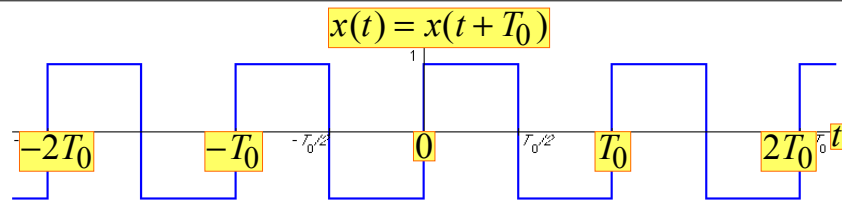
■ If  $x(t)$  is periodic with period  $T_0$ ,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

Therefore, since  $e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

## Square Wave Signal



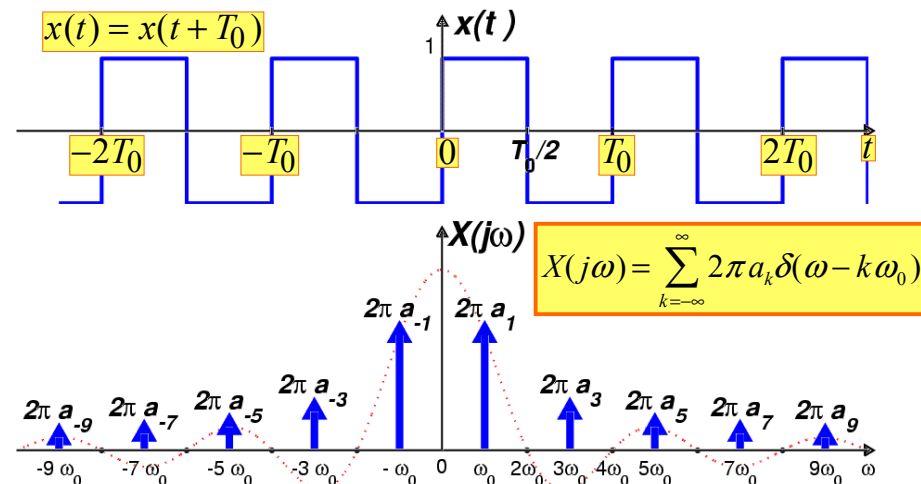
$$a_k = \frac{1}{T_0} \int_0^{T_0/2} (1) e^{-j\omega_0 k t} dt + \frac{1}{T_0} \int_{T_0/2}^{T_0} (-1) e^{-j\omega_0 k t} dt$$

$$a_k = \left. \frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0} \right|_0^{T_0/2} - \left. \frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0} \right|_{T_0/2}^{T_0} = \frac{1 - e^{-j\pi k}}{j\pi k}$$

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## Square Wave Fourier Transform



## Table of Easy FT Properties

### Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

### Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

### Frequency Shifting

$$x(t) e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

### Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$$

## Scaling Property

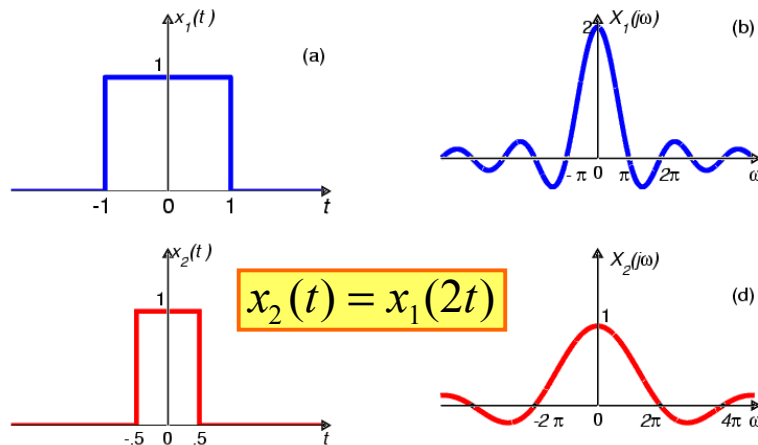
$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$$

$$\begin{aligned} \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt &= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau/a)} \left(\frac{1}{|a|}\right) d\tau \\ &= \frac{1}{|a|} X(j\frac{\omega}{a}) \end{aligned}$$

$$x(2t) \text{ shrinks; } \frac{1}{2} X(j\frac{\omega}{2}) \text{ expands}$$

## Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$$



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## Uncertainty Principle

- Try to make  $x(t)$  shorter
  - Then  $X(j\omega)$  will get wider
  - Narrow pulses have wide bandwidth
- Try to make  $X(j\omega)$  narrower
  - Then  $x(t)$  will have longer duration
- Cannot simultaneously reduce time duration and bandwidth**

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## Significant FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

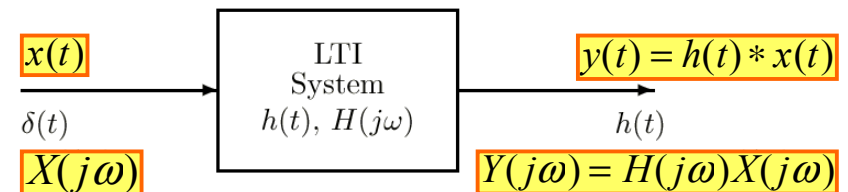
### Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

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## Convolution Property



- Convolution in the time-domain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

corresponds to **MULTIPLICATION** in the frequency-domain

$$Y(j\omega) = H(j\omega)X(j\omega)$$

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## Convolution Example

- Bandlimited **Input** Signal
  - “sinc” function
- Ideal LPF (Lowpass Filter)
  - $h(t)$**  is a “sinc”
- Output** is Bandlimited
  - Convolve “sincs”

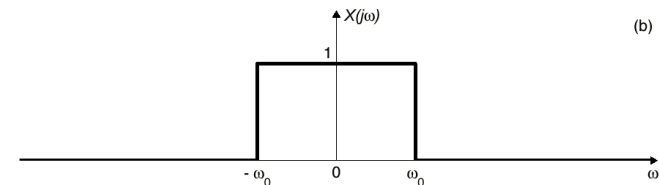
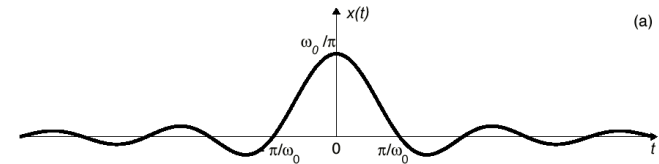
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## Ideally Bandlimited Signal

$$x(t) = \frac{\sin(100\pi t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$



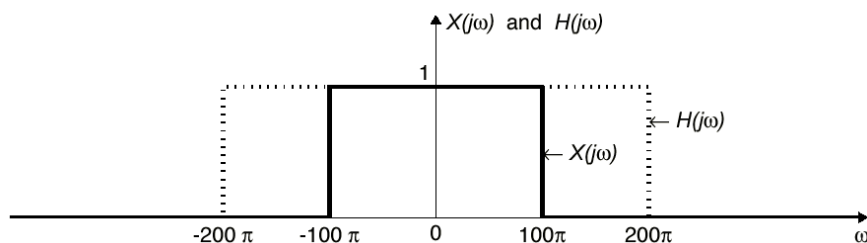
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## Convolution Example

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$



## Cosine Input to LTI System

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= H(j\omega)[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

$$= H(j\omega_0)\pi\delta(\omega - \omega_0) + H(-j\omega_0)\pi\delta(\omega + \omega_0)$$

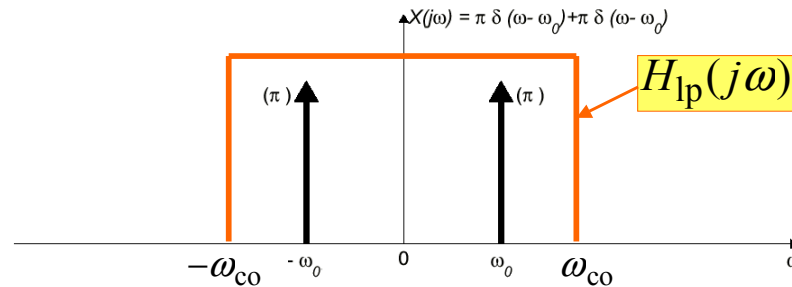
$$\begin{aligned} y(t) &= H(j\omega_0)\frac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\frac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\frac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\frac{1}{2}e^{-j\omega_0 t} \\ &= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) \end{aligned}$$

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## Ideal Lowpass Filter



$$y(t) = x(t) \quad \text{if } \omega_0 < \omega_{co}$$

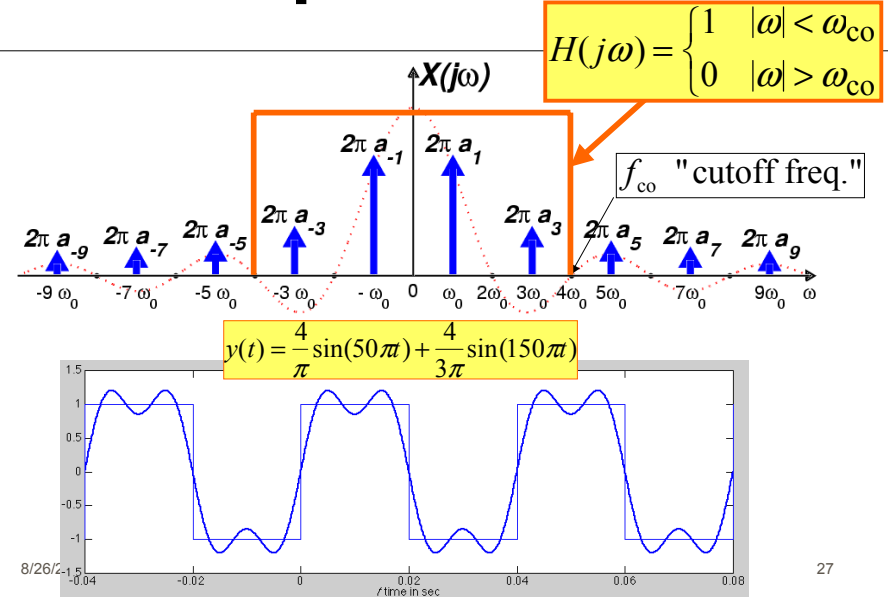
$$y(t) = 0 \quad \text{if } \omega_0 > \omega_{co}$$

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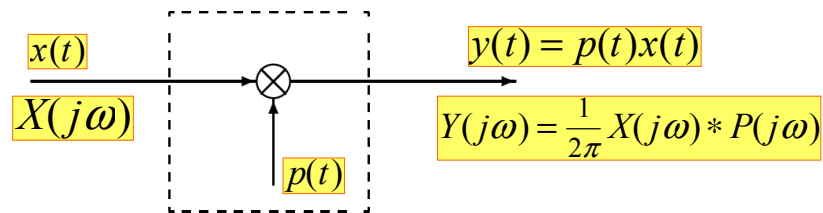
## Ideal Lowpass Filter



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## Signal Multiplier (Modulator)



- Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\omega - \theta)) d\theta$$

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## Frequency Shifting Property

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

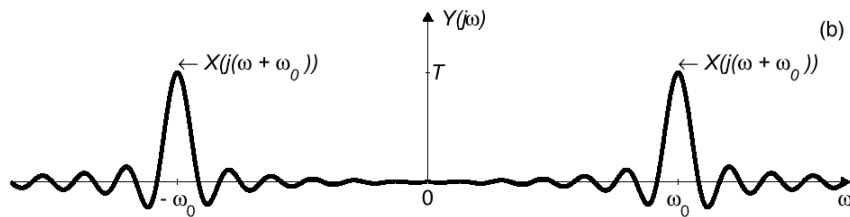
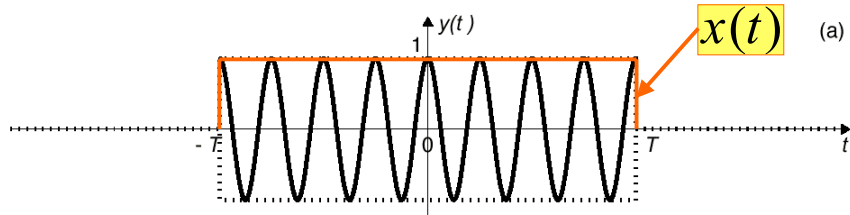
$$\int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt$$

$$= X(j(\omega - \omega_0))$$

$$y(t) = \frac{\sin 7t}{\pi t} e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0 - 7 < \omega < \omega_0 + 7 \\ 0 & \text{elsewhere} \end{cases}$$

$$y(t) = x(t) \cos(\omega_0 t) \Leftrightarrow$$

$$Y(j\omega) = \frac{1}{2} X(j(\omega - \omega_0)) + \frac{1}{2} X(j(\omega + \omega_0))$$



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## Differentiation Property

$$\frac{dx(t)}{dt} = \frac{d}{dt} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega) X(j\omega) e^{j\omega t} d\omega$$

Multiply by  $j\omega$

$$\frac{d}{dt} (e^{-at} u(t)) = -ae^{-at} u(t) + e^{-at} \delta(t)$$

$$= \delta(t) - ae^{-at} u(t)$$

$$\Leftrightarrow \frac{j\omega}{a + j\omega}$$

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